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Higher Order Perturbation Analysis of Plasma and Gravitational Waves

Hogere Orde Perturbatie Analyse van Plasma en Gravitationale Golven

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Chapter 1

Introduction

Plateau [1] initiated experimentally and theoretically the stability of a liquid cylinder with surface tension. Lord Rayleigh [2] improved this work and developed the linearized theory for sound waves (although the calculation of sound velocity goes back to Newton and Laplace) and for the Plateau experiments which was then applied to all kinds of fields: gravitation [3, 4, 5], plasma [6], (magneto)hydrodynamics [7, 8, 9], energy principles [10]. The linear theory flourished tremendously in the past century to a large extent due to the goal of fusion. Soon the need for nonlinear theories was manifest e.g. Callebaut [11, 12]. We therefore see in the previous decades a lot of work on nonlinear theory of plasma waves and instabilities being done. These yield either exact solutions or approximate ones. Often exact solutions are obtained after that the equations have been approximated. We may mention the papers with the exact solutions by Malfliet et al. [13, 14, 15], Hereman et al. [16, 17, 18], Verheest et al. [19, 20, 21] on solitary waves and those of Khater et al. on Bäcklund transformations and Painlevé analysis [22, 23, 24, 25]. Amiranashvili et al. [26] gave some exact solutions for standing waves in bounded plasmas without using the solitary wave theory but with some boundary conditions. Callebaut and Tsintsadze [27, 28] e.g., neglecting some higher order terms, dealt with the nonlinear bunching of Alfvén waves and the filamentation and modulation of weakly ionized magnetized plasmas. In fact, except for the approaches leading to solitary wave solutions, the nonlinear methods usually yield approximations and usually one has barely an idea how long these are valid in the behavior of the plasma. The approach used in this thesis exploits the Fourier analysis for nonlinear systems. It is rather different from the approaches just mentioned as it allows some insight in the convergence. Moreover, it gives useful results for the many cases where one can not find a closed form for the solitary waves. Indeed the solitary waves are an exceptional and rare case, comparable with a polynomial (as is clear e.g. from Malfliet's work, see references cited) while the general solution is an infinite series. The polynomial may use a function (e.g. tanh) instead of the (combined) variable itself. Similarly the series may use any function although the customary ones are exponentials and (co)sines. Infinite Fourier series may in principle be considered as an exact solution, but in practice it often is an approximation, which, however, allows clear insight on its validity.

The set of (partial differential) equations (e.g. equations (2.1) - (2.4)), together with some initial and/or boundary conditions, defines a set of functions (which are, of course, interrelated). From the Fourier theory it is known that if a periodic function is continuous from $-\infty$ to $+\infty$ and has a derivative which is piecewise monotonous and continuous, then the function may be developed in a Fourier series which is absolutely and uniformly convergent in any interval. In the thesis we deal with the single variable χ $(= \omega\, t + \boldsymbol{k} \cdot \boldsymbol{r})$, which is the combination of the four independent variables i.e. the angular frequency ($\omega = 2\pi\, \nu$, $\nu =$ frequency), the time (t), the wave vector ($\boldsymbol{k}$) and space ($\boldsymbol{r}$). Thus the conditions have to be satisfied for the function(s) of this combined variable. In particular the function should be periodic in $\omega\, t$ and in $\boldsymbol{k} \cdot \boldsymbol{r}$. However, an exponential growth is easily accommodated just like the periodic situations as was the case in a hydrodynamical problem [12]. Hence under rather general conditions it is possible to expand the functions defined by the set of equations. When will this breakdown? E.g. when the series diverges, i.e., physically speaking, when instability leading to disrupture develops meaning that a (large) amount of energy has been made available (either injected externally or freed by the system itself from e.g. its potential energy or, more generally, from its free energy). The convergence of the series puts conditions on the linear theory, mainly on its amplitude. In fact a linearized theory can never determine its own limitations: that has to be done by the nonlinear analysis.

In the previous works [12] it turned out that some experimental situations, in particular the oscillations and instabilities of a liquid jet, could be explained very well by this method. Moreover some cases appeared where the nonlinear theory showed that the linear theory was good even up to destruction of the configuration, while for wavelengths much larger than the diameter of the jet the nonlinear terms became dominant.

Another breakdown of the method may occur e.g. when the function(s) is (are) not periodic. However, in the linear perturbation theory one works usually with a periodic perturbation and this generates naturally higher order terms which are periodic too as is obvious e.g. in our present work and in the works of Callebaut [12, 29, 30]. For a non-periodic solution one has to take a wholly different start in the linear theory, e.g. by using a series in t and/or x, y, z, or some adequate combination of those, or if nevertheless a periodic or exponential start is used as first order term, to adapt profoundly the nonlinear terms. Such nonlinear approaches have been elaborated in various ways in the literature see e.g. the cited references of Malfliet et al.; Hereman et al.; Verheest et al.; Khater et al.; Callebaut and Tsintsadze.

1.1 The kind of nonlinear Fourier analysis

1.1.1 Callebaut's

Callebaut [12] developed a Fourier stability analysis going to second order or at most fourth order only in the cases of hydrodynamics (liquid jet with surface tension), magnetohydrodynamics (plasma cylinder pervaded with uniform magnetic field), gravitomagnetodynamics (spiral arm pervaded by uniform magnetic field). In the case of a liquid cylinder under the influence of surface tension the agreement with experiments [31] was excellent. The brief outline of the method is as follows (but for a detailed analysis of the method we refer to reference [30]):

(1) Consider only one Fourier component of the linearized perturbation theory and the nonlinear terms which are connected directly to that one component.

(2) Use $\epsilon_0 \exp \sigma t$ or $\epsilon_0 \cos \omega t$ (where σ is the growth rate and ω is the angular frequency) as amplitude, which means a fusion of the first order amplitude at $t = 0$, namely ϵ_0, with the time factor.

(3) In view of this and by iteration a hierarchy of coupled systems has been generated:

- System of zeroth order: equations of the equilibrium (index 0)

  Symbolic:

$$L_0 \begin{vmatrix} X_{10} \\ \dots \\ X_{n0} \end{vmatrix} = 0$$

  with L_0 an operator and X_i $(i = 1, 2, ..., n)$ the different relevant functions corresponding to e.g. the density, the pressure, the velocity,...

- System of first order: linearized theory (index 1)

  Symbolic:

$$L_1 \begin{vmatrix} X_{11} \\ \dots \\ X_{n1} \end{vmatrix} = 0$$

These equations provide the highly important dispersion relation, which means the relation between ω (or σ) and the wavenumber and equilibrium quantities..

- System of second order (index 2)

  Symbolic:

$$L_1 \begin{vmatrix} X_{12} \\ \dots \\ X_{n2} \end{vmatrix} = F_2(X_{i1}X_{j1})$$

  with the same L_1 as for the linearized system and with F_2 a function which is quadratic in X_{i1} $(i, j = 1, 2, ..., n)$.

- System of third order (index 3)

  Symbolic:

$$L_1 \begin{vmatrix} X_{13} \\ \dots \\ X_{n3} \end{vmatrix} = F_3(X_{i1}X_{j1}X_{k1}, X_{i2}X_{j1})$$

  again with the same L_1. F_3 is a function which is cubic in X_{i1} $(i, j, k = 1, 2, ..., n)$ and which contains also products of X_{i2} and X_{j1}.

- For the systems of higher order one only takes the particular solution into account.

Using the same basic method the nonlinear Fourier perturbation analysis for the case of infinite and homogeneous plasma media as well as gravitating media will be elaborated in this thesis. However, one has to note here that, while our analysis uses the combined variable χ $(= \omega t + \boldsymbol{k} \cdot \boldsymbol{r})$, the work of Callebaut on the other hand treated the four variables in χ independently.

1.1.2 Interference

As stated above, in the footsteps following Lord Rayleigh, the way to reach the dispersion relation is to linearize the system of equations and next to apply a Fourier analysis. This has the results that at least part of the equations is "algebraised" and that one may work with one single Fourier term, say $A\, e^{i(\omega t + \boldsymbol{k} \cdot \boldsymbol{r})}$. (In view of the boundary conditions it is often the case that one or two dimensions do not allow the Fourier analysis and that for those a differential equation has still to be solved. E.g. in the cylindrical case this leads to a Bessel function combined with the Fourier terms for the other directions). Clearly a complete linearized solution is then a sum or integral over such terms, still with arbitrary (but very small) amplitudes. Let the solution be say $\sum A_{\omega_j,\boldsymbol{k}_j}\, e^{i(\omega t+\boldsymbol{k}\cdot\boldsymbol{r})}$. We call this series the horizontal Fourier series. The substitution of it - in fact just one term is enough - yields the dispersion relation

$$\omega_j = f(k_j) \qquad \text{or simply} \qquad \omega = f(k)$$

in which equilibrium quantities occur too.

However, having derived the dispersion relation and sticking to one specific Fourier term with specific amplitude A and specific ω and $\boldsymbol{k}$, one may look at the higher order terms generated by this term by iteration in the nonlinear system: this leads to a series of terms of the type

$$a_j\, A^j\, e^{ij(\omega t+\boldsymbol{k}\cdot\boldsymbol{r})}$$

where the coefficients a_j are fixed by substitution in the nonlinear system. We may call this the vertical Fourier series. In fact it is the normal Fourier series of the function defined by the nonlinear system and by this first term $\left(A\, e^{i(\omega t+\boldsymbol{k}\cdot\boldsymbol{r})}\right)$. In fact this function (supposing it exists, i.e. it satisfies the very broad conditions required for a Fourier analysis to be valid which means essentially that it is periodic) is fully fixed as a solution of the system of equations and its first Fourier term which acts as a "kind of boundary or rather initial condition".

Clearly the horizontal and vertical Fourier series have a distinct character: the horizontal series has all arbitrary amplitudes and independent ω and $\boldsymbol{k}$ (however related by the dispersion relation) and they are all approximate solutions of the original system, satisfying the linearized equations only. The vertical series contains terms which have all multiples of the chosen specific ω and $\boldsymbol{k}$ and with amplitudes in well defined relationship to each other; they all together constitutes a family which represents on special but exact solution of the original system of partial differential equations.

So far there is nothing special, nothing nonlinear, about both Fourier series: the horizontal one is a solution of the linearized system; the vertical one happens to be a special solution of the nonlinear system.

Suppose now that we consider two "initial terms" from the linearized system: $A_1\, e^{i(\omega_1 t+\boldsymbol{k}_1\cdot\boldsymbol{r})}$ and $A_2\, e^{i(\omega_2 t+\boldsymbol{k}_2\cdot\boldsymbol{r})}$ where the pairs $(\omega_1, \boldsymbol{k}_1)$ and $(\omega_2, \boldsymbol{k}_2)$ are not multiples of each other. Substituting their sum in the system of equations leads to three parts a vertical Fourier series corresponding to A_1 a similar vertical Fourier series corresponding to A_2 and a mixed series. However, once the vertical series is obtained for one specific $A_1\, e^{i(\omega_1 t+\boldsymbol{k}_1\cdot\boldsymbol{r})}$ these three series are written down at once. In particular for the mixed series one has just to use combinatorial coefficients. The same is true when considering

j "initial terms". Clearly once we consider mixing (interference) of several initial terms we are working with a nonlinear analysis and this method may justly be called a kind of nonlinear Fourier analysis.

1.2 Brief note on plasmas

It is often stated that we live in less than 1% of the universe in which plasmas do not occur naturally. The remaining, say about 99%, of the observable universe is thought to be in the plasma state (we do not consider here the possibility of the unknown "dark matter" and "dark energy" which may constitute more than 90% of the universe according to some cosmological models). One encounters plasma in the things like the flash of a lightning bolt, the soft glow of the Aurora (Borealis for the northern hemisphere, Australis for the southern hemisphere), the conducting gas inside a fluorescence tube, various kinds of plasma chemistry and the slight amount of ionization in a rocket exhaust. In (or around) the Earth's atmosphere the ionosphere and the magnetosphere appear, relevant in various ways (radio transmission, capture of energetic rays, etc.). In addition, one may add the Van Allen radiation belt (i.e. englobing the two regions of high-energy-charged particles surrounding the Earth, the inner region centered at an altitude of 3200 km and the outer region at an altitude between 14,500 and 19,000 km). Outside the Earth's atmosphere, one encounters plasmas such as gaseous nebulae, stellar interiors and atmospheres, much of the interstellar hydrogen, solar wind (i.e. an emanation from the Sun's corona consisting of a flow of charged particles, mainly electrons and protons, that interacts with the magnetic field of the Earth and other planetary bodies). Although it is not readily apparent, plasmas affects us in one way or another. Thus it is interesting and important to study and hence have a thorough understanding of plasmas.

Various definitions of plasma (sometimes considered as a fourth state of matter) do exist, but in this thesis we define it as a *'quasineutral'* (i.e. neutral enough so that one can take $n_+ \approx n_- \approx n$, where n_+ and n_- are the particle densities of ions and electrons respectively and n is a common density called the plasma density, but not so neutral that all the interesting electromagnetic forces vanish) gas of charged and neutral particles which exhibits *'collective behaviour'* (i.e. "motions that depend not only on local conditions but on the state of the plasma in the remote regions as well") [32, pg. 3].

1.2.1 Plasma science

Plasma can occur whenever ordinary matter is heated to a temperature greater than about 10^4 degrees centigrade. The resulting plasmas are electrically charged gases or fluids. Hence the study of the state of matter which is ionized is called "plasma science." This includes "plasma physics", whose goal is to describe elementary processes in the matter which is sufficiently ionized (so that collective electromagnetic phenomena occur). Here plasma is normally described by Maxwell's equation for the electromagnetic fields and the Boltzmann equations to model the dynamics of electrons and ions. Simpler approximations, based on fluid descriptions for electrons and ions (e.g. magnetohydrodynamics), are also used.

Plasma science is central to the development of fusion (which is the source of energy of the Sun and the stars) as a fairly clean, renewable energy source. Great progress has been made towards this goal. For example fusion - plasma densities and confinement times as well as the achievable temperatures have been increased tremendously [33]. The growth rate is greater than the one achieved for computers [34]. Most of the remaining challenging questions are expected to be addressed in the coming international collaboration project, International Thermonuclear Experimental Reactor (ITER) [34]. Hence fusion is regarded as one of the candidates to our future energy sources which are compatible with our environment. However, of all viable energy options, fusion is characterized by exclusive properties, some of which represent distinct advantages (see below) over the other major sources of energy (e.g nuclear fission reactors, fossil fuels burning, etc).

1.2.2 Some of the advantages of fusion

The major advantages of fusion are as follows:

1. The fusion fuels (deuterium and tritium) are cheap and widely accessible.

2. Fusion is safe due to the fact that there is only a very small amount involved in each fusion reaction and the fusion process is not based on a neutron multiplication reaction. With any malfunction or incorrect handling the reaction will stop. An uncontrolled burn of fusion fuel, or explosion, is therefore excluded on physical grounds.

3. It does not pollute the atmosphere since no production of combustion gases as is the case for power plants burning fossil fuels. Moreover the radioactive waste is much smaller than for fission and, most of all, decays in a few years

For further advantages we suggest a paper by Ongena [35]. The advantages have been elaborated very extensively in this article.

1.3 Brief note on gravitation

It is understood that there are four fundamental forces (electromagnetism, gravitation, weak and strong nuclear forces) of nature. Of the four forces, gravitation is by far the weakest. This is the reason why it is less important to objects like people, particles etc. But it becomes important when astronomical bodies such as Earth, Moon, etc are involved. The gravitational forces are long range ones and hence can reach very far, as compared to electromagnetic forces which limit their range due to their tendency to quasi-neutrality. Before continuing, let us define gravitation. In reference [36], gravitation is defined as the universal force of attraction between all objects that tend to pull them toward one another. It affects all objects, all forms of matter and energy. It governs the motion of astronomical bodies such as moon, Earth etc. The complete mathematical theory of gravitation was developed by Sir Isaac Newton. Albert Einstein then formulated the more accurate theory of gravitation called general relativity. Thanks to the Earth's gravitation (i.e gravity) for holding objects on the surface of the Earth, otherwise we would have been sent floating off into space as a result of the Earth's spin. Note: Typically the word *gravitation* refers to the force in general, and the term *gravity* refers to the Earth's gravitational pull.

1.4 Plan of the thesis

The plan of the thesis is as follows: The thesis begins with this introductory chapter where we give a general overview of this thesis as well as of plasma and gravity. Then in chapter 2, we consider perturbations of ideal unmagnetized plasmas which are represented by systems of nonlinear partial differential equations. We then reduce each system to a nonlinear differential equation in one unknown, the combined variable $\chi = \omega t + \boldsymbol{k} \cdot \boldsymbol{r}$. The reduction process is done using either the first or the second procedure. These procedures reduce the system respectively to a second and a first order differential equations. By using *mathematica*, we solve the reduced equations and determine the coefficients for the higher orders quantities. This implies that, we have in the lowest order, terms proportional to $\exp(0)$, in first order proportional to $\exp(i\chi)$, in second one to $\exp(2i\chi)$, then $\exp(3i\chi)$ etc. Hence, in the lowest order, equilibrium properties are shown and integration constant (if any) can be fixed. In first order the dispersion relation (or relations) are found (which is nothing less than the linearizing procedure). In second order, due to nonlinearity, the generation of harmonics start: hence the coefficient (or coefficients in case one considers a medium, e.g. plasma, consisting of more than one kind of particle species) for the second order quantity can be found (being a function of the coefficient for the first order quantity). In third order the coefficient for the third order quantity can be found (being function of the coefficients for both first and second order quantities), etc.

In chapter 3 we again analyze the unmagnetized plasma waves. However, we now make some adjustments in the way we tackle integration involving potentials. This leads us into obtaining three "improved methods". We call them improved methods simply because one of the methods is the modification of the one considered in chapter 2. In addition, they are now producing exactly same results, without adding unwanted results, as it was the case previously. It is therefore the purpose of this chapter to: (a) indicate an improved method, which may shorten the calculating times drastically; (b) illustrate the improved method to the multi-species plasma; (c) apply the improved method to the electron - positron case; (d) compare the computer times required by the procedures (yielding the same results); (e) consider the case where our method gives bad convergence.

In the previous two chapters we neglected the magnetic contribution. We now, in chapter 4, study the theory for plasma including the magnetic effects due to currents created by the charged particles. Again, our particular interest is on the examination of the coefficients of higher order quantities and hence study the convergence issue.

In chapter 5 we make a nonlinear stability analysis on a (quasi-) infinite homogeneous gravitational medium in equilibrium. We use the Newtonian approximation with the exception that we introduce in the gravitational field equation, a cosmological constant so as to make the homogeneous equilibrium possible. We then apply the improved nonlinear Fourier analysis elaborated previously for plasmas. The obtained results are then compared with those of the plasma case.

As in chapter 5, we again in chapter 6, consider the medium infinite in all space dimensions and obeying the Newtonian law of gravitation to which a cosmological term is added. But now, as opposed to the analysis in chapter 5, we analyze inhomogeneous equilibria involving a varying gravitational potential.

In chapter 7 we consider an infinite, nonuniform gravitating medium in motion. Hence the initial velocity is now different from zero. Although formally the present analysis of the evolution of a dynamic inhomogeneous medium turns out to be formally quite similar to a previous analysis, the interpretation and some results are different.

In chapter 8 we again study the case of an infinite homogeneous gravitating medium (with a cosmological term) but now we suppose that the matter (which is assumed to be perfectly conducting) is being pervaded by a homogeneous magnetic field. The Alfvén waves are included here, however, seen from a different angle. Three different situations occur here (a) incompressible (purely Alfvén waves; gravitation irrelevant); (b) compressible (without gravitation) (c) compressible with gravitation.

We summarize the conclusions of this research in chapter 9 (Summary) and in chapter 10 (Nederlandse samenvatting).

Chapter 2

Perturbation Analysis for Unmagnetized Plasma Waves I

2.1 Introduction

This chapter gives first the basis (section 2.2), from which we formulate a model, state the basic equations and consider the equilibrium case. In section 2.3 we study the case of unmagnetized cold plasma when ions stand still. Here the perturbation analysis using exponentials as well as cosines is performed. The analysis begins with the reduction of the system of partial differential equations to one equation by using two different procedures. Then, the determination of several coefficients to higher orders follows. From the determined coefficients, one obtains the analytic expressions and then study the convergence of the series involved, both analytically and graphically. In section 2.4 the case of unmagnetized electron plasma is investigated. Here the analysis similar to that considered in section 2.3 is performed. But now convergence is studied by using graphical method only, since it is difficult to study it analytically. We finish the chapter by giving the concluding remarks in section 2.5.

Various alternative procedures for the simple case like cold plasma are elaborated because one wants to find out which procedure is the simplest and the safest.

2.2 Basis

2.2.1 Model

We consider a multi-species plasma in a medium infinite in all directions (no boundary conditions) and at rest, i.e. we consider an equilibrium configuration in which (not necessarily small) perturbations are generated. No source terms are considered (no particle creation or recombination). Viscosity, resistivity, gravitation and the magnetic contribution are neglected.

2.2.2 Basic equations

The basic equations are the continuity equations, the equations of motion, the Poisson equation and the polytropic equations. Thus, the basic system of equations can be written:

$$\partial_t n_\alpha + div(n_\alpha \boldsymbol{v}_\alpha) = 0, \tag{2.1}$$

$$n_\alpha m_\alpha \frac{d\boldsymbol{v}_\alpha}{dt} = -\boldsymbol{\nabla} p_\alpha - q_\alpha n_\alpha \boldsymbol{\nabla}\varphi, \tag{2.2}$$

$$\Delta\varphi = -\frac{1}{\varepsilon}\sum_\alpha (q_\alpha n_\alpha), \tag{2.3}$$

$$p_\alpha = K_\alpha n_\alpha^{\Gamma_\alpha}, \tag{2.4}$$

where n_α, q_α, $\boldsymbol{v}_\alpha$, m_α, p_α, K_α and Γ_α are respectively the number density, charge density, velocity, mass, pressure, polytropic constant and polytropic exponent of the $\alpha-$th kind of particle species (neutral particles have $q = 0$ of course), φ is the electrical potential and ε is the permittivity, whose value in vacuum is 8.85×10^{-12} C/Vm.

2.2.3 Equilibrium

At equilibrium, we consider plasma to be at rest. Thus using a subscript zero to indicate quantities at equilibrium, we put:

$$\boldsymbol{v}_{\alpha 0} = \boldsymbol{v}_0 = 0 \text{ and } \boldsymbol{E}_0 = -\boldsymbol{\nabla}\varphi_0 = 0, \tag{2.5}$$

where $\boldsymbol{E}$ is the electric field. From (2.5) we see that φ_0 is an arbitrary constant. Without loss of generality, we choose it to be equal to zero. Plugging these into our basic system of equations we get:

$$\partial_t n_{\alpha 0} = 0, \tag{2.6}$$

$$\nabla p_{\alpha 0} = 0, \tag{2.7}$$

$$p_{\alpha 0} = K_\alpha n_{\alpha 0}^{\Gamma_\alpha}, \tag{2.8}$$

$$\sum_\alpha (q_\alpha n_{\alpha 0}) = 0. \tag{2.9}$$

From equations (2.6) - (2.8) we see that $n_{\alpha 0}$ and $p_{\alpha 0}$ are constants which are independent of both time and space. Equation (2.9) expresses the quasi-neutrality ($\sum_\alpha q_\alpha n_\alpha = 0$).

2.2.4 Some assumptions

The basic system of equations (section 2.2.2) is for plasma consisting of α kinds of species. But for simplicity from now onwards we shall deal with plasma consisting of positive particles (usually ions) and negative particles (usually electrons), unless otherwise stated. We shall also assume that there is only one type of ion species in the plasma and thus put the index $\alpha = +$, $-$. Here the signs " $+$ " and " $-$ " are used systematically for positive ions (with charge $q_+ = e = 1.6 \times 10^{-19}$ C and $m_+ = 1.67 \times 10^{-27}$ kg for proton) and electrons (having charge $q_- = -e$ and mass $m_- = 0.91 \times 10^{-30}$ kg). And due to quasineutrality, we put $n_{+0} = n_{-0} = n_0$.

We further assume that the phase velocity of the wave (i.e. ω/k) is the same for all terms. Physically speaking there is no dispersion between the terms of one family generated by a solution of the linearized equation as far as the phase velocity is concerned. All phases travel with the same speed, however, nothing is said about the group velocity. This assumption is very plausible as the linearized solution generates higher orders with a multiple of χ $(= \omega t + \boldsymbol{k} \cdot \boldsymbol{r})$. (In fact it are integer multiples, which makes the full solution actually periodic if the first order term is periodic, which is usually, but not always the case). We then make the natural supposition that all quantities are functions of χ alone. This makes it possible to write

$$\partial_t Y \equiv \omega \partial_\chi Y = \omega Y' \quad \text{and} \quad \nabla Y \equiv \boldsymbol{k} \partial_\chi Y = \boldsymbol{k} Y', \tag{2.10}$$

where the accent means the derivative with respect to χ and Y stands for either $n_\pm$, $\boldsymbol{v}_\pm$ or φ. With the above information, one is then ready to analyze various cases. In order to test our procedures we consider, in the next two main sections, the cases of plasma when (a) pressure is neglected and (b) pressure is included.

2.3 One component cold plasma

2.3.1 Analysis of perturbation using exponentials

Here we suppose that the kinetic motion of the charged particles is negligible and thus the temperature is low enough to be approximated by zero in consideration with other effects (in the present case electric ones). And since ions are much heavier than electrons, we introduce the approximation $n_+ = n_0$, meaning that the ion density is unperturbed. (The case where ions also are perturbed will be considered later in chapter 3). Our basic system of equations then reduces to

$$\partial_t n_- + div(n_- v_-) = 0, \tag{2.11}$$

$$n_- m_- \frac{dv_-}{dt} = e n_- \nabla \varphi, \tag{2.12}$$

$$\Delta \varphi = \frac{e}{\varepsilon}(n_- - n_0). \tag{2.13}$$

Applying (2.10) to equations (2.11) - (2.13) we have

$$\omega n'_- + \boldsymbol{k} \cdot (n_- \boldsymbol{v}_-)' = 0, \tag{2.14}$$

$$m_- (\omega + \boldsymbol{v}_- \cdot \boldsymbol{k}) \boldsymbol{v}'_- = e\boldsymbol{k}\varphi', \tag{2.15}$$

$$k^2 \varphi'' = \frac{e}{\varepsilon}(n_- - n_0). \tag{2.16}$$

We then eliminate $\boldsymbol{v}_-$ and φ from (2.14) - (2.16) in order to reduce the system to a single equation. This is accomplished as follows: Integrate (2.14) to obtain

$$(\omega + \boldsymbol{k} \cdot \boldsymbol{v}_-) n_- = \in_- \text{ (a constant)}. \tag{2.17}$$

At equilibrium, this gives $\in_- = \omega n_{-0} = \omega n_0$. Hence equation (2.18) becomes

$$(\omega + \boldsymbol{k} \cdot \boldsymbol{v}_-) n_- = \omega n_0. \tag{2.18}$$

Then write equation (2.14) as

$$\boldsymbol{k} \cdot \boldsymbol{v}'_- = -\frac{(\omega + \boldsymbol{k} \cdot \boldsymbol{v}_-) n'_-}{n_-}.$$

Substituting (2.18) into this we get

$$\boldsymbol{k} \cdot \boldsymbol{v}'_- = -\frac{\omega n_0 n'_-}{n_-^2}. \tag{2.19}$$

Take dot product with $\boldsymbol{k}$ on both sides of (2.15) to have

$$m_- (\omega + \boldsymbol{k} \cdot \boldsymbol{v}_-) \, \boldsymbol{k} \cdot \boldsymbol{v}'_- = ek^2 \varphi'.$$

Substituting equations (2.18) and (2.19) into this we obtain

$$\varphi' = -\frac{m_- \, \omega^2 \, n_0^2 \, n'_-}{e \, k^2 \, n_-^3}. \tag{2.20}$$

We then finish our process of reducing the system to a single equation by using one of the following two procedures:

Procedure 1: (Reduction to a nonlinear differential equation of second order)

Differentiate (2.20) with respect to χ to obtain

$$\varphi'' = \frac{m_- \omega^2 \, n_0^2}{ek^2 n_-^4} \left(3n_-'^2 - n_- n_-'' \right). \tag{2.21}$$

Substituting (2.21) into (2.16) and rearranging we have

$$\boxed{3n_-'^2 - n_- n_-'' + \tfrac{\omega_-^2 \, n_-^4}{\omega^2 \, n_0^3} \left(n_0 - n_- \right) = 0,} \tag{2.22}$$

where $\omega_-^2 = (e^2 \, n_0)/(\varepsilon \, m_-)$ is the square of the electron plasma frequency.

Procedure 2: (Reduction to a nonlinear differential equation of first order)

Multiply φ' on both sides of equation (2.16) and then integrate to have

$$\varphi'^2 + C_1 = \frac{2e}{\varepsilon k^2} \int (n_- - n_0) \, \varphi' d\chi,$$

which yields, after substituting (2.20) into it, the following equation:

$$\left(-\frac{m_- \, \omega^2 \, n_0^2 \, n'_-}{ek^2 n_-^3} \right)^2 + C_1 = \frac{m_- \, \omega^2 \, n_0^2}{\varepsilon k^4 n_-^2} \left(2n_- - n_0 \right). \tag{2.23}$$

We then determine C_1 by substituting the lowest order of n_- into this equation. Hence, rearranging (2.23) after substituting for C_1, we obtain

$$\boxed{n_-'^2 = \tfrac{-\omega_-^2}{\omega^2 \, n_0^4} (n_- - n_0)^2 n_-^4} \tag{2.24}$$

and $\omega^2 = 0$ which gives the trivial solution.

Comment:
The two nonlinear equations i.e. (2.22) and (2.24) have to be equivalent (taking into account initial conditions). Equation (2.22) is a differential equation of second order while equation (2.24) is of first order. *It is our purpose to show that they lead indeed to the same results and to find out which procedure is the easiest for future use.*

Equation (2.24) can be integrated and hence obtain a fully integrated equation. But one faces the problem of determining the constant of integration. This may lead one into making some subtle assumptions in order to obtain results. Thus we do not consider it as one of our procedures. For mathematical illustration of this procedure we briefly consider it in appendix A.

2.3.1.1 Determination of some values of some physical quantities

(I) Particle density

As we explained in section 2.2.4, the solution will be periodic if the solution of the linearized equation is periodic, which is the commonly accepted situation. We also know that it is possible for a periodic motion to be decomposed by Fourier analysis into a superposition of sinusoidal oscillations with different wavelengths and frequencies. If the oscillation amplitude is small, then the waveform may be approximated as sinusoidal and one may work with only one component. Hence we let a single Fourier term to be our first perturbation and we put

$$n_{-1} = A\, n_0 e^{i\chi}, \tag{2.25}$$

where A (considered in this thesis to be real quantity though in general it is a complex quantity) is the initial amplitude of the first order perturbation (relative to the equilibrium density). Later on, the upper limit of A or radius of convergence of the series will be determined; this is the maximum value for which our series development can be valid. The general form of the particle density associated with the first order term (2.25) can, in view of our considerations, be written as

$$n = n_0 + n_{-1} + n_{-2} + \cdots = n_0 \sum_{j}^{N} a_{-j\rho}\, A^j\, e^{ij\chi}, \tag{2.26}$$

where N goes to infinity and $a_{-0\rho} = a_{-1\rho} = 1$. Notice that we have used here ρ as one of the subscripts. In this notation it is not a function. Hence from now on-wards, all ρ's, v's, φ's and f's appearing in the subscripts, should be considered as symbols and not as functions. They are used there just to indicate the quantity to which the coefficients do belong. For example, $a_{-1\rho}$

is the coefficient in the first order term of the negative (normally electrons) particle density expression (cold plasma case) and c_{-1_ρ} is the coefficient in the first order term of the negative particle density when the case of two components plasma is considers.

Other values of a_{-j_ρ}, where $j = 2, 3, \cdots$, are determined by substituting equation (2.26) into either (2.22) or (2.24) and solve. So, in the lowest order, we have terms proportional to exp(0), in first order proportional to $\exp(i\chi)$, in second one to $\exp(2i\chi)$, then $\exp(3i\chi)$ etc. Hence in the lowest order, equilibrium properties are shown and integration constant (if any) can be fixed. In first order the dispersion relation (or relations) are found (which is nothing less than the linearizing procedure). In second order, due to nonlinearity, the generation of harmonics start: hence a_{-2_ρ} can be found (being a function of $a_{-1_\rho} = 1$). In third order a_{-3_ρ} can be found (being a function of $a_{-1_\rho} = 1$ and a_{-2_ρ}), etc. The mathematical illustrations of this is given below:

Determination of a_{-j_ρ}'s using equation (2.22)

1. First order: We put equation (2.26) into equation (2.22) and linearize to obtain
$$n''_{-1} + \frac{\omega_-^2}{\omega^2} n_{-1} = 0.$$
Then substituting for n_- this and rearranging yields the angular frequency of oscillation of the electrons
$$\boxed{\omega^2 = \omega_-^2.} \tag{2.27}$$
This is the well-known dispersion relation relation for cold plasma. Hence ω_- is the frequency by which electrons will oscillate around the position of the ions.

2. For the second order, insert equation (2.26) into (2.22) to obtain
$$3n'^2_{-1} - (n_0 + n_{-1})n''_{-1} - n_0\, n''_{-2} - \frac{\omega_-^2}{\omega_-^2}\left[(n_{-1} + n_{-2})n_0 + 4n^2_{-1}\right] = 0.$$
Thus, substituting for n_{-1}, n_{-2} and ω^2 we get $a_{-2_\rho} = 2$.

Following these steps, we wrote a program (not included in the thesis) in *mathematica* [37] to solve for the remaining values of a_{-j_ρ}. In table 2.1

we show some of the obtained values of $a_{-j\rho}$. Examining these values very closely, we infer the following analytic expression

$$\boxed{a_{-j\rho} = \frac{j^{j-1}}{(j-1)!} = \frac{j^j}{j!}.} \tag{2.28}$$

Putting (2.28) into (2.26) we get

$$\boxed{n_- = n_0 \left(1 + \sum_{j=1}^{N} \frac{j^j}{j!} A^j e^{ij\chi}\right),} \tag{2.29}$$

which converges when $|A| < \frac{1}{e} \approx 0.367879\cdots$, according to d'Alembert ratio test [38, p. 45].

j	0	1	2	3	4	5	6	7	8	9	10
$a_{-j\rho}$	1	1	2	$\frac{9}{2}$	$\frac{32}{3}$	$\frac{625}{24}$	$\frac{324}{5}$	$\frac{117649}{720}$	$\frac{131072}{315}$	$\frac{4782969}{4480}$	$\frac{1562500}{567}$

Table 2.1: *A table showing values of* $a_{-0\rho}$, $a_{-1\rho}$, $\cdots$, $a_{-10\rho}$.

Determination of $a_{-j\rho}$'s using equation (2.24)

1. In this case it is not possible to determine the dispersion relation by using the linearization procedure as we did previously. Thus, the first nontrivial results can only be obtained from the terms in $A^2\exp(2i\chi)$. So substituting (2.26) into (2.24) one obtains:

   $$\omega^2 n'^2_{-1} = -\omega_-^2 n_{-1}^2.$$

   Using (2.25) into this yields equation (2.27). Notice that all terms involving a_{-2} disappeared from the above equation.

2. The next step is to obtain the coefficients in second order. Here one substitutes (2.26) (up to order three) into (2.24) to have

   $$\omega^2 n_0 n'_{-1}(n'_{-1} + 2n'_{-2}) = -\omega_-^2 n_{-1} \left[(n_{-1} + 2n_{-2})n_0 + 4n_{-1}^2\right],$$

   from which we get $a_{-2\rho} = 2$ after substituting for ω^2 and n_{-1}. Though we used equation (2.26) up to order three, but one noticed here that all terms involving a_{-3} disappeared.

3. To determine $a_{-3\rho}$, we again substitute (2.26) (but now up to order four) into (2.24) and solve. Notice here that all terms involving a_{-4} will not appear and hence we determine the value of $a_{-3\rho}$ after substituting for ω and the known $a_{-j\rho}$'s.

Other values for $a_{-j\rho}$'s in higher orders are determined in a similar manner. The determined values confirm those in table 2.1.

<u>What will happen if one takes square roots on equation (2.24)?</u>

Taking square roots on both sides of equation (2.24) yields

$$\boxed{n'_- = \pm\sqrt{\frac{-\omega_-^2}{\omega^2 n_0^4}}\,(n_- - n_0)\, n_-^2.} \tag{2.30}$$

The minus sign under the square root is harmless since n_- will be replaced by equation (2.26). Let us determine some values of the coefficients $a_{j\chi}$'s:

1. First order: Plugging (2.26) into (2.30) and linearizing we obtain

$$\omega = \pm\omega_- \tag{2.31}$$

   which is consistent with equation (2.27).

2. Determination of $a_{-2\rho}$: Putting (2.26) (up to order two) in (2.30) we get:

$$1 + 2a_{-2\rho}\, A\, e^{i\chi} = \pm\frac{\omega_-}{\omega}\left(1 + (2 + a_{-2\rho})Ae^{i\chi}\right). \tag{2.32}$$

   Substituting equation (2.31) into this equation we obtain $a_{-2\rho} = 2$.

3. For third order, after substituting for $a_{-1\rho} = 1$ and $a_{-2\rho} = 2$, yields

$$1 + A\, e^{i\chi}\left(4 + 3a_{-3\rho}\, A\, e^{i\chi}\right) = \pm\frac{\omega_-}{\omega}\left\{1 + A\, e^{i\chi}\left(2 + a_{-3\rho}\, A\, e^{i\chi}\right)\right.$$
$$\left. +2A\, e^{i\chi}\left(1 + 4A\, e^{i\chi}\right) + A^2 e^{2i\chi}\right\}, \tag{2.33}$$

   from which we get $a_{-3\rho} = 9/2$.

Following these steps, we wrote a program in *mathematica* to solve for the remaining values of $a_{-s\rho}$. Again the determined values confirm those given in table 2.1.

(II) Velocity

The velocity terms are investigated by solving equation (2.18) as elaborated below:
We substitute for n_- and $\boldsymbol{v}_-$ in (2.18) and linearize the resulted equation to get

$$\boldsymbol{k} \cdot \boldsymbol{v}_{-1} n_0 + \omega n_{-1} = 0. \tag{2.34}$$

Our perturbed (oscillating) quantities are assumed to behave sinusoidally: hence in view of equation (2.26) we let

$$\boldsymbol{v}_- = \boldsymbol{v}_{-1} + \boldsymbol{v}_{-2} + \boldsymbol{v}_{-3} + \cdots = \frac{\boldsymbol{k}}{k} \sum_{j=0}^{N} a_{-jv} A^j e^{ij\chi}, \tag{2.35}$$

where $a_{-0v} = 0$ while other a_{-jv}'s are constants to be determined. Then by substituting for n_{-1} and $\boldsymbol{v}_{-1}$ in (2.34) we obtain

$$a_{-1v} = -\frac{\omega}{k}.$$

We again substitute for n_- and $\boldsymbol{v}_-$ in (2.18), but this time up to second order. On expanding the obtained equation and neglecting quantities involving orders higher than two, we have

$$a_{-2v} = -\frac{\omega}{k}.$$

j	0	1	2	3	4	5	6	7	8	9	10
a_{-jv}	0	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{125}{24}$	$\frac{54}{5}$	$\frac{16807}{720}$	$\frac{16384}{315}$	$\frac{531441}{4480}$	$\frac{156250}{567}$

Table 2.2: *A table of values of* $a_{-0v}, a_{-1v}, \cdots, a_{-10v}$, *in unit* $-\omega/k$.

Other values of a_{-jv}'s can be determined following the similar procedure. In table 2.2, we give the first ten a_{-jv}'s obtained using a program written in *mathematica* in order to determine the values of a_{-jv}'s. From these values, we inferred the analytic expression

$$\boxed{a_{-jv} = -\frac{j^{j-1}}{j!}\frac{\omega}{k},} \tag{2.36}$$

so that a_{-jv} has the dimensions of a velocity. We then put this into (2.35) and tested for convergence using d'Alembert's test. We found out that, the series is convergent when $|A| < e^{-1} \approx 0.367879\cdots$, in harmony with the convergence of the series for the particle density.

(III) Potential

Here we integrate (2.20) with respect to χ to get

$$\varphi + C_3 = \frac{m_-\omega^2 n_0^2}{2ek^2 n_-^2}. \tag{2.37}$$

By substituting the lowest order quantities into this we determine C_3 to be

$$C_3 = \frac{m_-\omega^2}{2ek^2}.$$

Hence equation (2.37) becomes

$$\varphi = \frac{m_-\omega^2 n_0^2}{2ek^2}\left(\frac{1}{n_-^2} - \frac{1}{n_0^2}\right). \tag{2.38}$$

j	0	1	2	3	4	5	6	7	8	9	10
$a_{j\varphi}$	0	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{25}{24}$	$-\frac{9}{5}$	$-\frac{2401}{720}$	$-\frac{2048}{315}$	$-\frac{59049}{4480}$	$-\frac{15625}{567}$

Table 2.3: *A table showing values of* $a_{0\varphi}, a_{1\varphi}, \cdots, a_{10\varphi}$, *in unit* $(en_0)/(k^2\varepsilon)$.

In view of equation (2.26) we let

$$\varphi = \sum_{j=0}^{N} a_{j\varphi} A^j e^{ij\chi} \tag{2.39}$$

with $a_{0\varphi} = 0$. We then substitute this into (2.38) and solve for $a_{j\varphi}$ using one of the *mathematica* programs to obtain results seen in table 2.3. From these results, we infer the following general expression:

$$\boxed{a_{j\varphi} = -\frac{j^{j-2}}{j!}\frac{en_0}{k^2\varepsilon}.} \tag{2.40}$$

Thus $a_{j\varphi}$ has dimensions of an electric potential. As in the previous analysis, the series in (2.39) with (2.40) converges when $|A| < e^{-1} \approx 0.367879\cdots$ in harmony with the convergence of the series for the particle density and the velocity.

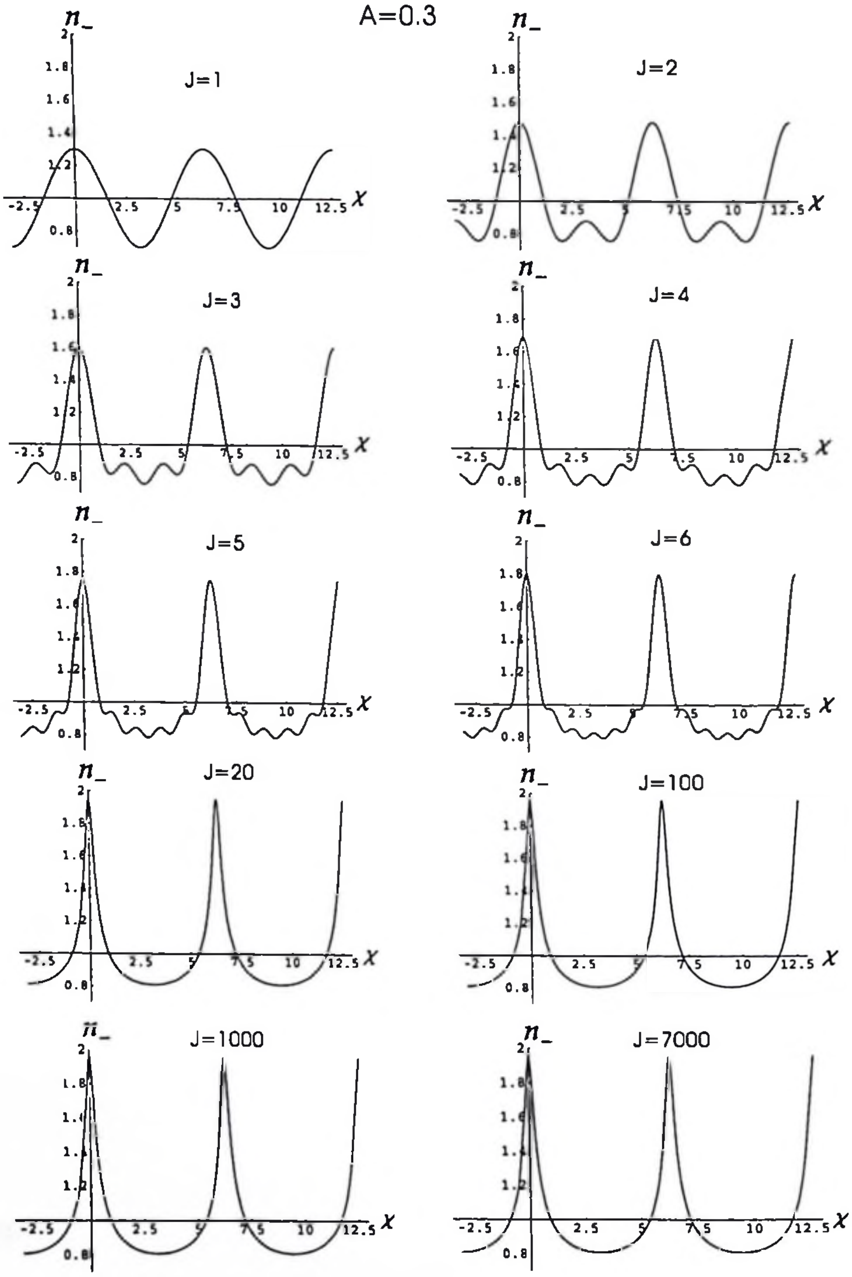

Figure 2.1: *The convergence case: Graphs of* χ *(*$=\omega t + \boldsymbol{k}\cdot\boldsymbol{r}$*) against* n_- *(particle density) for various values of* j *(the number of terms taken into account) with* n_0 *(the particle density at equilibrium) being scaled to one and* A *(the amplitude) being put equal to* 0.3. *For the convergent series (e.g. when* $A = 0.3$ *or any value less than* $e^{-1} \approx 0.367879\cdots$*), all values of* n_- *(in the graphs) are always positive (based on the numerical calculations up to order 7000).*

2.3.1.2 Graphical results

Convergence and divergence cases

The analytical results showed that our series will only converge if $|A|$ is less than $1/e \approx 0.367879\cdots$. In order to visualize this let us sketch some graphs involving $|A| < 1/e \approx 0.367879\cdots$ and some involving $|A| > 1/e \approx 0.367879\cdots$.

Convergence case

This is the case where the series involves A less than $e^{-1} \approx 0.367879\cdots$. For the illustration purposes, we sketch here some graphs involving $|A| = 0.3$ (see figure 2.1).

On the figure we have graphs of χ against n_- for the amplitude $A = 0.3$ and various values of j (the number of terms taken into account) with n_0 (the particle density at equilibrium) being scaled to one. So, when $j = 1$, we located the global minimum is at $(\chi, n_-) = (3, 0.7)$ and global maximum at $(0, 1.3)$. When $j = 2$ (i.e. when two terms are involved) we have non-linearity and hence observe the start of harmonics generation. So, we see extra extrema being produced in the graph. The new minimum and maximum points are found to be at points $(2, 0.76)$ and $(0, 1.48)$ respectively. For $j = 3$, at $(3, 0.75)$ and $(0, 1.6)$ respectively. At $j = 10$ (graph omitted), n_- has minimum at 0.8 and maximum at 1.9. And when $j = 20$, we have for the minimum and maximum points, n_- equal to 0.8 and 1.9 respectively. The same value is obtained for the higher orders. To be certain, we determined graphs up to when $j = 7000$ (though one could have stopped at say order 1000) and found the same result. In short, from these graphs we observed the following:

(a) All values of n_- are always positive. This is based on the numerical calculations of up to order 7000.

(b) For the minimum points, the values of the ordinate (in this case n_-) tend to run away from zero as the value of j increases. The value of n_- then stop varying when j reaches a certain point (in this case it stopped varying when $j = 10$).

(c) Graphs are bounded (in this case between 0.7 and 2).

The characteristics similar to those indicated above, were observed when we sketched graphs involving other $|A|$'s which are less than $1/e \approx 0.367879\cdots$. It follows then that, our series are convergent when $|A| < 1/e \approx 0.367879\cdots$. Before making a final remark, let us consider the other side of the coin by sketching some graphs involving $|A| > 1/e \approx 0.367879\cdots$.

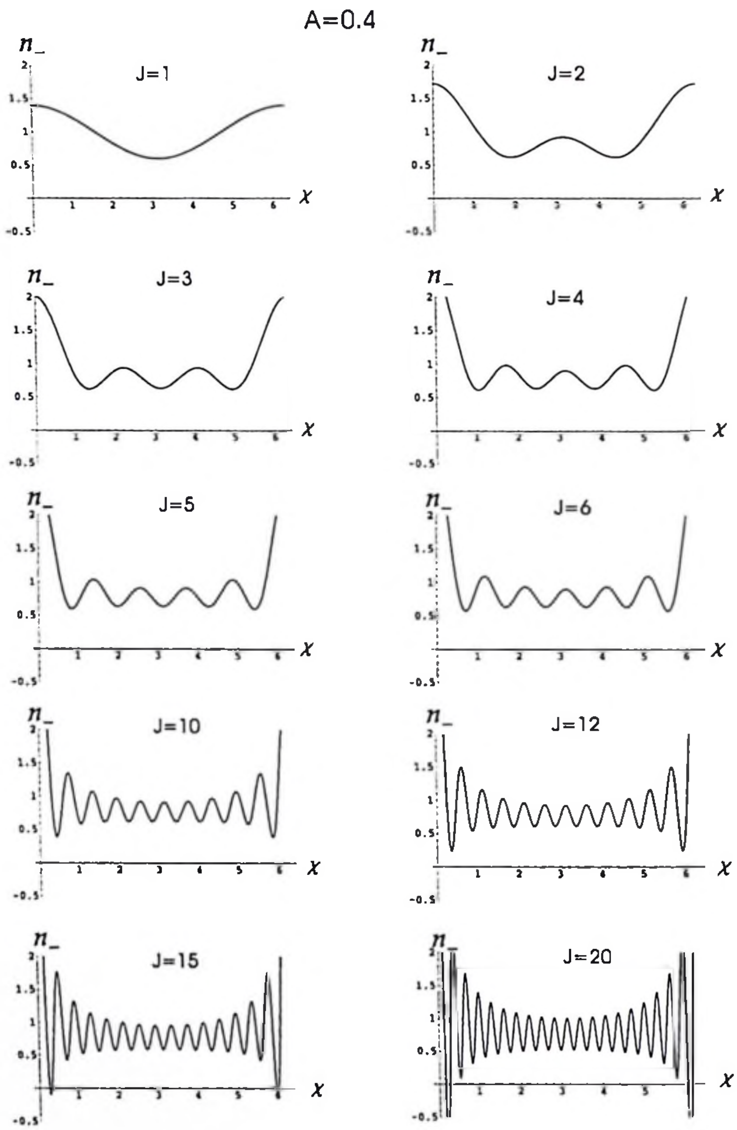

Figure 2.2: *The divergence case: Graphs of* χ $(= \omega t + \boldsymbol{k} \cdot \boldsymbol{r})$ *against* n_- *(particle density) for various values of* j *(the number of terms taken into account) with* n_0 *(the particle density at equilibrium) being scaled to one and the amplitude* $A = 0.4$*. For divergent series (i.e. when the series involves* A *greater or approximately equal to* $e^{-1} \approx 0.367879 \cdots$*) the graphs give some negative values in* n_-*, which is physically prohibited.*

Divergence case

This is the case when the series involves A greater or approximately equal to $e^{-1} \approx 0.367879\cdots$.
So, as we did above we again sketch graphs of χ against n_- for various values of j with n_0 being scaled to one but now we put $A = 0.4$ (taken as an example $|A|$ having values greater than $1/e \approx 0.367879\cdots$). Some of the sketched graph in are given in figure 2.2.

From the graphs we note that when $|A| > 1/e \approx 0.367879\cdots$, some points in higher orders are below zero (e.g. the case of $j = 15$ and above). Negative density is excluded because of the conservation of particles and charge and because of the positive uniform background.

We also notice from the graph that, instead of the values of ordinate (n_-) in the minimum points to move away from zero (as was the case for the convergent case), they rush toward it. For example, when $j = 1$, the ordinate in the minimum point is 0.6. And for the case of $j = 10$ and $j = 15$, the values of the minimum n_- are respectively 0.4 and -0.1. In this case we do not even need more than 15 terms before getting a negative n_-, which is physically prohibited. We therefore consider the series in this case to be divergent.

Once the limit of convergence is exceeded the divergency occurs even by using few terms as we have just seen for $A = 0.4$. Additionally, by looking at the trend of the decreasing values of n_- as j increases, give an indication of divergence.

To complete this part of divergence, let us see what happens when $|A|$ is approximately equal to $1/e \approx 0.367879\cdots$. In this case we sketch some graphs with $A = 0.37$. So from figure 2.3 (I) we notice that for $j = 100$, we still have minimum value which are still positive. But when we $j = 200$ or above, the minimum values of n_- become negative. In figure 2.3 (II) we zoom in to have a clear visualization of the negative values of some n_-'s. Hence, as we have negatives in n_-, then our series will diverge if $A = 0.37$. For the case of $A = 0.368$ (a value much closer to $0.367879\cdots$), one needs a lot of terms to have a negative n_- (illustrated by $j = 5000$ - graphs omitted).

Remarks We therefore conclude that, if at least one of the minimum points is negative, then the series is divergent for all values greater or equal to the chosen A. But if the minimum points for n_- are positive then the series is convergent for A equal or less than the selected value. It is nice to see that these observations agree with the result of the analytic considerations. In fact this suggests a graphical method for the determination of the convergence limits (e.g. those in table 2.8), which will be needed in all cases where the convergence can not be determined analytically.

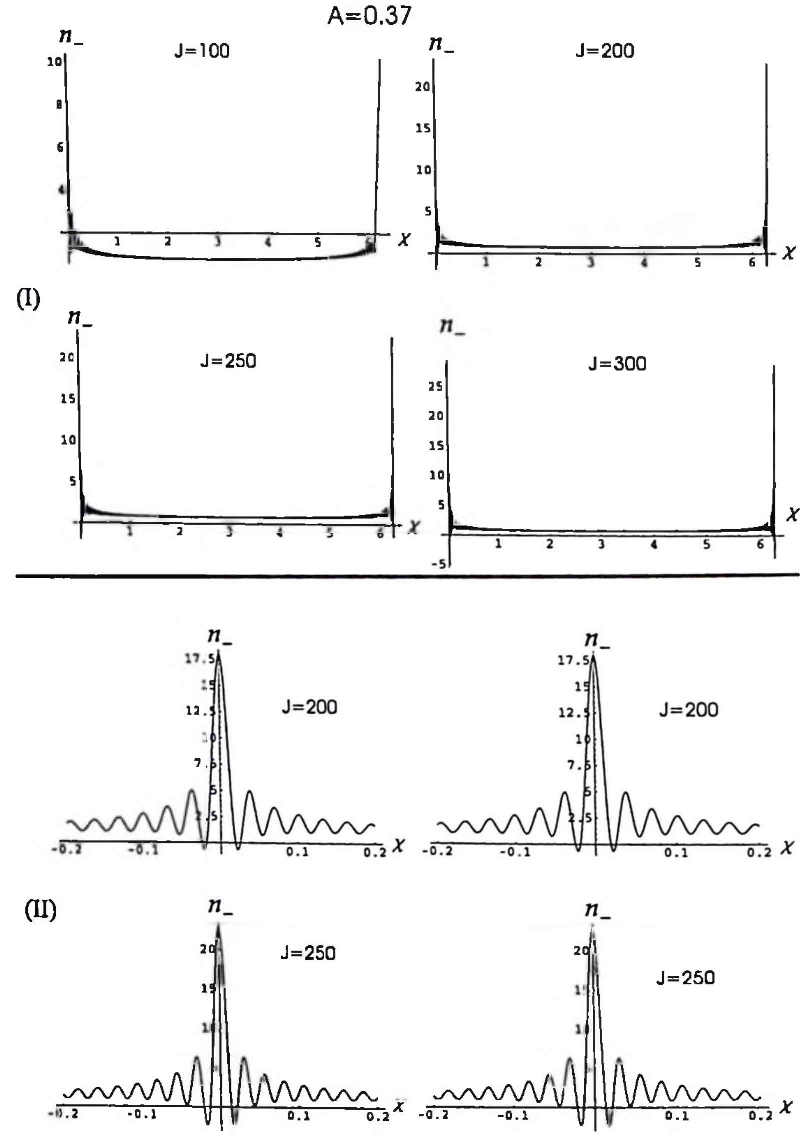

Figure 2.3: *Some of the graphs of* χ *vs* n_- *which were used in the estimations of the limits of convergence. Here* $A = 0.37$, *the values of* j *are as indicated on the graphs; with* n_0 *being scaled to one. Part (II) gives the zoom in parts of specific areas in order to see clearly the negative value of* n_- *for the cases* $j = 200$ *and* $j = 250$.

Note: For cold plasma the convergence is independent of k, in contrast to the situation with pressure (to be considered in section 2.4) or the cases treated by Callebaut [12, 29, 30].

Conservation of particles

We also calculated the area under the curve of $n_- - n_0$ against χ (only real values of n_- in (2.26) are being considered) and found out that for the first order case, the area is zero. When higher orders are included in the series, the area obtained ranged between 1×10^{-14} and 1×10^{-16}. As the equilibrium density n_0 was normalized to unity, this shows that the number of particles is conserved within the computational error.

Comparison with solitary waves

The wave under consideration, i.e. the solution (2.29) may be considered as a kind of solitary wave as is visible on figure 2.1 too. The differences with the 'more traditional' solitary waves are the following:

1. Periodicity: We recover the same wave when adding to x any multiple of the distance $\pm 2\pi/k$. Hence along the x- as we have an infinity of identical travelling waves (and similarly along the time axis). The more traditional solitary waves (usually expressed by means of a polynomial in $\tanh a(x \pm vt)$ (a and v constant) are one travelling wave. (cf. Malfliet et al. [13, 14, 15]). However, in nature a row of such travelling waves does occur (some ocean water e.g.) which rather resemble the periodic wave under study.

2. Related to the periodicity: The present wave does not vanish at $\pm\infty$. The more traditional solitary waves vanish either at one end or at both ends ($\pm\infty$).

3. The present wave has various maxima, showing a wave character (with various amplitudes) while the more traditional solitary wave is usually one lump.

4. The present wave has not the soliton character in the usual sense: adding two waves requires additional terms (interference). However, if the periods are commensurable, the same interference pattern may be repeated, which is a kind of soliton behaviour. Some traditional solitary waves have soliton character, some have not.

5. The wave (2.29) is the solution of equation (2.22) or equation (2.24). However, considering equations (2.1) - (2.4), we deal not only with the density, but the density, the velocity, the pressure, the potential are all linked together, Although each of these quantities may be considered individually as a solitary wave it is the combination of them which corresponds to the physical situation. The same situation does not always occur for the more traditional solitary waves.

2.3.1.3 Energy

Here we consider the convergence of the series for potential energy ($pe = [en_{+0} + (-en_-)]\varphi = (n_0 - n_-)e\varphi$), kinetic energy ($ke = m_- n_- v_-^2/2$), total energy (i.e. $energy = pe + ke$), $\int_0^{2\pi}(pe \times pe^*)d\chi$, $\int_0^{2\pi}(ke \times ke^*)d\chi$ and $\int_0^{2\pi}(energy \times energy^*)d\chi$, where pe^*, ke^* and $energy^*$ are the complex conjugates. Each of these series converges with $|A| < 1/e \approx 0.367879$. One can generally show this by considering two series $a = a_1A + a_2A^2 + a_3A^3 + \ldots + a_NA^N + \cdots$ and $b = b_1A + b_2A^2 + b_3A^3 + \ldots + b_NA^N + \cdots$ which have properties that $\frac{a_{N+1}}{a_N} = e \pm \kappa_N$ and $\frac{b_{N+1}}{b_N} = e \pm \tau_N$ respectively. Here κ_N and τ_N are small quantities converging to zero when $N \to \infty$. From $\frac{a_{N+1}}{a_N} = e \pm \kappa_N$ one gets $a_2 = a_1(e \pm \kappa_1)$, $a_3 = a_2(e \pm \kappa_2) = a_1(e \pm \kappa_1)(e \pm \kappa_2)$ and so on. Similarly for b_{N+1}. We then take the product of the series a and b. From the product, we divide the term involving A^3 by that involving A^2 to get

$$\frac{a_1b_1A^3\left[(e \pm \tau_1) + (e \pm \kappa_1)\right]}{a_1b_1A^2} = A[2e \pm (\text{peanuts})].$$

A	Potential energy	Kinetic energy	Total energy
0.001	1.0025	0.5015	1.5040
0.01	1.0256	0.5154	1.5410
0.1	1.3294	0.7041	2.0335
0.2	1.9730	1.13335	3.1063
0.25	2.6122	1.5903	4.2024
0.3	3.9366	2.6060	6.5426

Table 2.4: *A table showing various values of energies densities (in unit $\beta = m_- n_0 \omega^2 A^2/k^2$) for various A. The energies increase steadily with the amplitude.*

If we divide terms involving A^4 by the one involving A^3 we obtain

$$A\left(\frac{3e^2 \pm e(\text{peanuts})}{2e \pm (\text{peanuts})}\right).$$

Again dividing terms involving A^N by those containing A^{N-1} we get

$$A\left(\frac{(N-1)e^{N-2} \pm e(\text{peanuts})}{(N-2)e^{N-3} \pm e(\text{peanuts})}\right).$$

In the limit this gives $|Ae|$ which implies that, for our series to converge it is required that $|A| < 1/e \approx 0.367879\cdots$, just like the cases considered previously.

For visualization purposes, we calculated *pe*, *ke* and energy for some A when $\chi = 0$. We summarize the result in table 2.4. From the table we see that the virial theorem for equilibrium ($2\,ke = pe$) is approximately satisfied especially when $A \leq 0.1$ as it should as the deviation from equilibrium is smaller when A is small. With this result and the convergence of the series considered before, we conclude here that the expressions for the particle density, velocity and electrical potential verify the energy conservation law.

2.3.1.4 Discussion

- In section 2.3 we considered two procedures in order to complete the reduction process of our system. Of the two procedures one notes that procedure 1 is simpler and safer than the other one. This means that a second order equation is easier to handle than an equivalent first order one. To some extent we may understand this because using a higher order differential equation is usually more condensed than any of its integrated forms. Nevertheless it is surprising. However it is a lucky aspect as for more complicated equations it may be very difficulty to obtain integrated forms. However we shall see later that using certain simplifications may affect the outcome, i.e. which procedure is the fastest.

- From the analysis of the results obtained for cold plasma we note that the series yields results and converges but only for amplitudes which involve less than 37% of the equilibrium density. The convergence for the cold plasma is independent of any of the quantities relevant to the problem (k, ω, ω_-). This contrasts with the case of plasma when pressure is included (see below). It contrasts too with the cases treated by Callebaut [12, 29, 30] with e.g. a marked dependence on k and which suited experiments well.

- A sketch of (ω, k) graph for $\omega^2 = \omega_-^2$ is a straight line passing through $(0, \omega_-)$ and parallel to the k-axis. This means ω does not depend on k. Hence the group velocity $\partial\omega/\partial k$ is zero, implying that cold plasma oscillations are not propagating as is well known. This is so because the kinetic motions of the charged particles is neglected.

- From the graphical results we note that, there is an agreement between the graphical and analytical results on the issues of convergence and divergence. This match encouraged us to use the graphical procedure in approximating the radius of convergence in the situations where it was difficult to determine such value analytically. In addition the graphical method was used in showing, within the computational error, the conservation of the number of particles. Hence basing on the advantages mentioned above, we shall use the graphical method in approximating the radius of convergence of our series, whenever we fail to determine it analytically.

- It is suggested that the theory, in particular the convergence, should be verified in a (cold) plasma by setting up very strong perturbations caused by an externally applied field. Maybe this can be done in a very quiet plasma like in a Q-machine (Quiescent Plasma Machine) having a magnetized alkaline plasma or unmagnetized argon plasma of a DP-Machine (Double Plasma Machine). Both machines are in use for example in the laboratory of Prof. R. Schrittwieser (Innsbruck University, Innsbruck, Austria). We discussed with Schrittwieser on the possibility of setting such experiment in his laboratory [39].

2.3.2 Analysis of perturbation using cosines

Here we address the question: what happens if we use cosines (or sines) instead of exponentials?

Let us use (2.22) in the determination of various coefficients with (2.26) being replaced by

$$n_- = n_0 + n_{-1} + n_{-2} + \cdots = n_0 + \sum_{l=1}^{N+1} \sum_{j=l}^{N} n_0\, a_{-j,l\rho}\, A^j \cos(l\,\chi), \qquad (2.41)$$

where N goes to infinity and $a_{-j,l\rho}$ are constants to be determined. With the help of a *mathematica* program we obtained results given in tables 2.5. The first value in this table (i.e. $a_{-1,1\rho}$) must be different from zero otherwise there will be no perturbation. Considering coefficients of expressions of higher orders we find out that all $a_{-j,1\rho}$'s (where $j = 2, 3, \cdots$) are arbitrary constants. These correspond to waves independent of our choice of the first order term. Hence, in view of reducing the length of our calculations, we decided to put all $a_{-j,1\rho}$'s equal to zero. With the exception of these values, other quantities in the table were purely determined using the programs written for that purpose.

j	1	2	3	4	5	6	7	8	9	10
$a_{-j,1\rho}$	1	0	0	0	0	0	0	0	0	0
$a_{-j,2\rho}$		1	0	$\frac{-1}{12}$	0	$\frac{-5}{192}$	0	$\frac{-259}{23040}$	0	$\frac{-6197}{1105920}$
$a_{-j,3\rho}$			$\frac{9}{8}$	0	$\frac{-27}{128}$	0	$\frac{-153}{2560}$	0	$\frac{-1011}{40960}$	0
$a_{-j,4\rho}$				$\frac{4}{3}$	0	$\frac{-2}{5}$	0	$\frac{-7}{72}$	0	$\frac{-451}{12096}$
$a_{-j,5\rho}$					$\frac{625}{384}$	0	$\frac{-3125}{4608}$	0	$\frac{-34375}{258048}$	0
$a_{-j,6\rho}$						$\frac{81}{40}$	0	$\frac{-243}{224}$	0	$\frac{-81}{512}$
$a_{-j,7\rho}$							$\frac{117649}{46080}$	0	$\frac{-823543}{491520}$	0
$a_{-j,8\rho}$								$\frac{1024}{315}$	0	$\frac{-1024}{405}$
$a_{-j,9\rho}$									$\frac{4782969}{1146880}$	0
$a_{-j,10\rho}$										$\frac{390625}{72576}$

Table 2.5: *A table showing some of the values of $a_{-j,l\rho}$: this is the coefficient of $n_0 A^j \cos(l\chi)$ in the series for electron density.*

For the velocity and the potential we use $v_- = \sum_{l=1}^{N+1} \sum_{j=l}^{N} a_{-j,lv} A^j \cos(l\chi)$ and $\varphi = \sum_{l=1}^{N+1} \sum_{j=l}^{N} a_{j,l\varphi} A^j \cos(l\chi)$ in (2.18) and (2.38) respectively. Again, with the assistance of a *mathematica* program we obtained results given in tables 2.6 and 2.7.

The results in tables 2.5 - 2.7 are then compared with respect to results given in tables 2.1 - 2.3. Doing so one notes that the coefficients $a_{-j,jf}$ (with

j	1	2	3	4	5	6	7	8	9	10
$a_{-j,1v}$	-1	0	$\frac{3}{4}$	0	$-\frac{17}{96}$	0	$\frac{115}{1536}$	0	$\frac{-319}{23040}$	0
$a_{-j,2v}$		$-\frac{1}{2}$	0	$\frac{11}{24}$	0	$\frac{-43}{384}$	0	$\frac{697}{15360}$	0	$\frac{-20123}{2211840}$
$a_{-j,3v}$			$-\frac{3}{8}$	0	$\frac{51}{128}$	0	$\frac{-69}{640}$	0	$\frac{1583}{40960}$	0
$a_{-j,4v}$				$-\frac{1}{3}$	0	$\frac{2}{5}$	0	$\frac{-59}{480}$	0	$\frac{9281}{241920}$
$a_{-j,5v}$					$-\frac{125}{384}$	0	$\frac{125}{288}$	0	$\frac{-19625}{129024}$	0
$a_{-j,6v}$						$-\frac{27}{80}$	0	$\frac{1107}{2240}$	0	$\frac{-7047}{35840}$
$a_{-j,7v}$							$\frac{-16807}{46080}$	0	$\frac{285719}{491520}$	0
$a_{-j,8v}$								$\frac{-128}{315}$	0	$\frac{1984}{2835}$
$a_{-j,9v}$									$\frac{-531441}{1146880}$	0
$a_{-j,10v}$										$\frac{-78125}{145152}$

Table 2.6: *A table showing some of the values of $a_{-j,lv}$ which are coefficients of $A^j \cos(l\chi)$ in the expression of velocity in unit ω/k.*

j	1	2	3	4	5	5	7	8	9	10
$a_{-j,1\varphi}$	-1	0	$\frac{3}{2}$	0	$\frac{-15}{8}$	0	$\frac{307}{128}$	0	$\frac{-9359}{3072}$	0
$a_{-j,2\varphi}$		$-\frac{1}{4}$	0	$\frac{19}{48}$	0	$\frac{-379}{768}$	0	$\frac{58219}{92160}$	0	$\frac{-3548731}{4423680}$
$a_{-j,3\varphi}$			$-\frac{1}{8}$	0	$\frac{27}{128}$	0	$\frac{-673}{2560}$	0	$\frac{41353}{122880}$	0
$a_{-j,4\varphi}$				$-\frac{1}{12}$	0	$\frac{3}{20}$	0	$\frac{-1081}{5760}$	0	$\frac{46441}{193536}$
$a_{-j,5\varphi}$					$\frac{-25}{384}$	0	$\frac{575}{4608}$	0	$\frac{-40625}{258048}$	0
$a_{-j,6\varphi}$						$\frac{-9}{160}$	0	$\frac{513}{4480}$	0	$\frac{-2097}{14336}$
$a_{-j,7\varphi}$							$\frac{-2401}{46080}$	0	$\frac{55223}{491520}$	0
$a_{-j,8\varphi}$								$\frac{-16}{315}$	0	$\frac{328}{2835}$
$a_{-j,9\varphi}$									$\frac{-59049}{1146880}$	0
$a_{-j,10\varphi}$										$\frac{-15625}{290304}$

Table 2.7: *A table showing some of the values of* $a_{j,l\varphi}$. *These constants, in unit* $(e\,n_0)/(k^2\,\varepsilon)$, *are the coefficients of* $A^j\cos(l\chi)$ *in the expression of the potential.*

f here standing for either ρ, v or φ) are related to a_{-jf} through a formula

$$\boxed{a_{-j,jf} = \frac{a_{-jf}}{2^{j-1}}.} \tag{2.42}$$

For other values (the non-diagonal ones) in tables 2.5 - 2.7, one may infer some general formulae for example $a_{-j,(j-2)f} = -\frac{(j-2)^{j-1}(j-3)!}{2^j(j-1)!(j-4)!}$, with $j = 4, 5\cdots$.

Checking for convergence of (2.41) with (2.42) (putting $f = \rho$), one finds out that the series converges with $|A| < 2/e \approx 0.735758\cdots$. The same applies when f is replaced by v or φ. In fact $e^{j\,i\chi} = \cos j\,\chi + i\sin j\,\chi$ consists physically of a sum of two waves. Using only one (say the cosine e.g.) reduces the wave and its energy by a factor two, thus allowing the critical amplitude for convergence to be doubled.

2.4 One component plasma with pressure

2.4.1 Analysis of perturbation when exponentials are used

Here we assume that the kinetic motion of electrons is not negligible and thus the pressure of electrons is present. Hence our basic system of equations becomes equations (2.11), (2.13) together with

$$n_- m_- \frac{dv_-}{dt} = -\nabla p_- + e n_- \nabla\varphi, \tag{2.43}$$

and

$$p_- = K_- n_-^{\Gamma_-}. \tag{2.44}$$

After specifying our system of equations, we then start our analysis by reducing the system into a single equation as indicated below:

2.4.1.1 Elimination procedures

We once more let all quantities be functions of χ alone. Hence as we did previously in section 2.3, we obtain from the above system

$$\varphi' = \frac{m_-}{e\, n_-^3\, k^2}\left(\frac{k^2\, v_{s-}^2\, n_-^{\Gamma_-+1}}{n_0^{\Gamma_--1}} - \omega^2 n_0^2\right) n'_-, \tag{2.45}$$

or

$$\varphi' = \frac{m_- n_0^2\,(\omega_{vn}^2 - \omega^2)\, n'_-}{e\, n_-^3\, k^2}, \tag{2.46}$$

where $\omega_{vn}^2 = k^2\, v_{s-}^2\, (n_-/n_0)^{\Gamma_-+1}$ and $v_{s-}^2 = K_- \Gamma_-\, n_0^{\Gamma_--1}/m_-$ is the sound velocity of electrons. Equation (2.46) is the same as (2.20) with the pressure term added. We therefore use (2.46) in the following procedures in order to complete the reduction process in the same way as before.

<u>Procedure 1</u> (Reduction to a nonlinear differential equation of second order)

Differentiate (2.46) with respect to χ and substitute into (2.16) to get

$$\left(\frac{k^2\, v_{s-}^2 (\Gamma_- - 2)\, n_-^{\Gamma_-+1}}{n_0^{\Gamma_--1}} + 3\,\omega^2 n_0^2\right) n'^2_- + \left(\frac{k^2\, v_{s-}^2\, n_-^{\Gamma_-+1}}{n_0^{\Gamma_--1}} - \omega^2 n_0^2\right) n_-\, n''_-$$

$$= \frac{\omega_-^2\, n_-^4 (n_- - n_0)}{n_0} \tag{2.47}$$

or

$$\boxed{\left[(\Gamma_- - 2)\omega_{vn}^2 + 3\omega^2\right] n'^2_- + \left(\omega_{vn}^2 - \omega^2\right) n_- n''_- = \omega_-^2 n_-^4 (n_- - n_0)/n_0^3.} \tag{2.48}$$

This is the equation to be used in one of the *mathematica* programs in order to determine b_{-sp}'s as we shall see later.

Procedure 2 (**Reduction to a nonlinear differential equation of first order**)

Here we repeat what we did in procedure 2 of section 2.3 but we replace φ' in that procedure by (2.46) to obtain

$$\left(\frac{k^2 v_{s-}^2 n_-^{\Gamma_-+1}}{n_0^{\Gamma_--1}} - \omega^2 n_0^2\right)^2 n_-'^2 = \frac{2\,\omega_-^2\, n_-^4}{n_0}\left\{\frac{k^2 v_{s-}^2 n_-^2}{\Gamma_-\,(\Gamma_- - 1)\, n_0^{\Gamma_--1}}\right.$$
$$\left.\times\left[\Gamma_- n_-^{\Gamma_--1}(n_- - n_0) - (n_-^{\Gamma_-} - n_0^{\Gamma_-})\right] - \frac{\omega^2 n_0^2}{2n_0}(n_- - n_0)^2\right\} \qquad (2.49)$$

or

$$\boxed{(\omega_{vn}^2 - \omega^2)^2\, n_0^5\, n_-'^2 = 2\,\omega_-^2\, n_-^4\left\{D_n\, k^2 v_{s-}^2 n_-^2 - [\omega^2 n_0 (n_- - n_0)^2/2]\right\},} \qquad (2.50)$$

where $D_n = \left[\Gamma_- n_-^{\Gamma_--1}(n_- - n_0) - (n_-^{\Gamma_-} - n_0^{\Gamma_-})\right] / \left(\Gamma_-\,(\Gamma_- - 1)\, n_0^{\Gamma_--1}\right)$. Equation (2.50) may be used equally well as (2.48) in the determination of n_-. Once n_- is obtained we may fix the potential. We illustrate this mathematically as follows: first integrate (2.46) with respect to χ to get

$$\varphi + C_e = \frac{m_- n_0^2}{e k^2 n_-^2}\left[\frac{\omega_{vn}^2}{(\Gamma_- - 1)} + \frac{\omega^2}{2}\right]. \qquad (2.51)$$

Then determine the constant C_e by substituting the lowest order quantities (i.e. $n_- = n_0$ and $\varphi = \varphi_0 = 0$) into equation (2.51). Putting the obtained expression for C_e into (2.51) and rearranging yields

$$\boxed{\varphi = m_-\left\{\left[D_{vn}/(\Gamma_- - 1)\right] + \left[\omega^2(n_0^2 - n_-^2)/2\right]\right\}/(k^2 e n^2),} \qquad (2.52)$$

where $D_{vn} = (n_0^2\omega_{vn}^2 - k^2 v_{s-}^2 n_-^2)$. If one neglects the term involving pressure terms (or putting $v_{s-} = 0$) one recovers equation (2.38) seen previously in section 2.3.1.1. Using equation (2.52), we obtain the coefficients of the higher order terms in the potential.

2.4.1.2 Dispersion Relation

Here we begin by replacing $a_{-j\rho}$'s in (2.26) by $b_{-j\rho}$ in order to avoid confusion. Then, using a *mathematica* program with N being replaced by 1, we determine from equation (2.48) the dispersion relation:

$$\boxed{\omega^2 = \omega_-^2 + k^2 v_{s-}^2,} \qquad (2.53)$$

which modifies equation (2.27) by adding the terms due to pressure. This equation is the (Langmuir) dispersion relation for electron plasma waves [40, p.146]. Note that equation (2.53) is slightly more general than the usual Langmuir equation as it uses Γ_- instead of γ_-, the ratio of specific heats.

To determine the dispersion relation using equation (2.50), one is forced to include the second order quantities (not linearized quantities as in the case of equation (2.48)). We therefore put (2.26) with $N = 2$ into (2.50) and solve. Doing so we obtain the dispersion relation (2.53) together with

$$\omega^2 = k^2 v_{s-}^2. \tag{2.54}$$

But the latter is the dispersion relation for sound waves and it was not supposed to appear here. Due to this fact we reject it here. The reason for the cropping up of this additional result, which we call "false result", could be the multiplication of φ' during the integration process (cf procedure 2 in section 2.3). May be the multiplication added some terms which in turn contribute in producing our false result.

Comments

Let us sketch the graphs of the dispersion relation (2.53) together with the oblique asymptote (dispersion relation for sound waves). In addition, on the same axes we sketch the graph for the dispersion relations for the cold plasma case (see figure 2.4). We then put k_m to be the value of the wavenumber at the point of intersection of graphs for equations (2.27) and that of the oblique asymptote. If $k << k_m$ (long wavelengths), then electric forces dominate and the plasma frequency ω_- of the electrons is a fair approximation. This is due to the fact that the electrostatic potential has a long range and hence becomes relatively more important for longer wavelengths than the pressure variations. When $k >> k_m$ (short wavelengths), then the approximation for the sound waves dominates and the convergence decreases. If $k = k_m$, then we have the wavenumber at the approximated point of transition between the cold plasma region and the sound dominated region.

The expression for k_m is given by:

$$k_m = \frac{\omega_-}{v_{s-}} = \frac{\sqrt{e^2 n_0/\varepsilon m_-}}{\sqrt{p_{-0}\Gamma_-/n_0 m_-}} = e\sqrt{\frac{n_0}{\varepsilon \Gamma_- k_B T}}, \tag{2.55}$$

where we have put $p_{-0} = n_0 k_B T$. In fact we have to use T_-, but we assume that $T = T_- = T_+$, although it happens that T_- is two orders of magnitude larger than T_+.

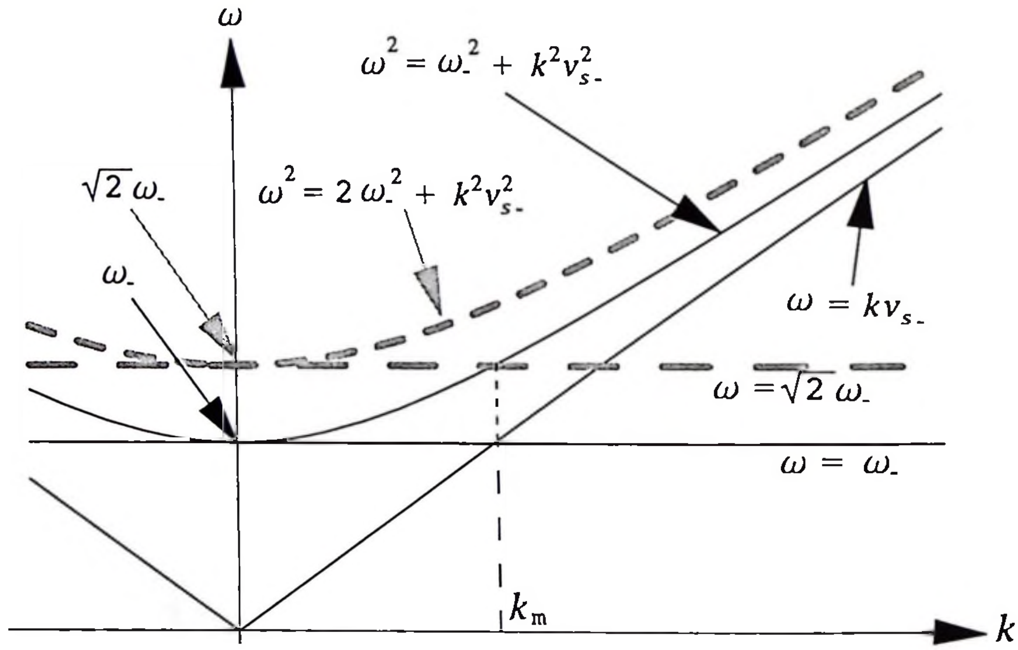

Figure 2.4: *The (ω, k) graphs for the dispersion relations for cold plasma ($\omega^2 = \omega_-^2$), sound waves ($\omega^2 = k^2 v_{s-}^2$)and plasma with electron pressure ($\omega^2 = \omega_-^2 + k^2 v_{s-}^2$). (Note that we sketched mainly one branch because $\pm k$ are not really distinct). The straight lines passing through the origin correspond to pure sound waves and are the asymptotes of the hyperbola. For comparison purposes, this figure gives also the graphs of $\omega^2 = 2\omega_-^2$, $\omega^2 = 2\omega_-^2 + k^2 v_{s-}^2$ (heavy dashed lines) and that of sound waves, which were obtained when the electron-positron plasma case was considered.*

2.4.1.3 Determination of coefficients of the higher order terms

Using equation (2.53) in (2.48) we solved, with the assistance of *mathematica* program of course, equation (2.48) up to order fifteen and determined $b_{-j\rho}$'s (see the first ten $b_{-j\rho}$'s in appendix B.1.1). Similar results are obtained if one puts (2.53) into (2.50). The obtained $b_{-j\rho}$'s are then substituted into (2.26) to give an expression for the particle density up to order j.

If one solves (2.50) with (2.54), one gets $b_{-j\rho}$'s which depend on $\exp(i\chi)$ and this is a contradiction as $b_{-j\rho}$'s are constants. This can be taken as an additional reason for rejecting equation (2.54), i.e. it leads one into getting the "false $b_{-j\rho}$'s". We thus reject also all $b_{-j\rho}$'s obtained when this expression was used.

For the coefficients in the velocity and potential we again evaluated constants b_{-jv} and $b_{-j\varphi}$ by using the appropriate *mathematica* programs written specifically to solve for the required coefficients. The summaries of some of the obtained coefficients for velocity and potential are respectively found in appendices B.1.2 and B.1.3.

Electron plasma case (ions stand still)		
ω_Ω	k	$\lvert A\rvert$
0	0 ($v_{s-} = 0$ or $k_m = \infty$)	$< e^{-1}$
1/2	$\frac{\sqrt{3}}{4}\, k_m$	$< 0.775e^{-1}$
1	$\frac{\sqrt{3}}{2\sqrt{2}}\, k_m$	$< 0.612e^{-1}$
8/3	k_m	$< 0.353e^{-1}$
8	$\sqrt{3}\, k_m$	$< 0.163e^{-1}$
800	$10\sqrt{3}\, k_m$	$< 0.0019e^{-1}$

Table 2.8: *Plasma with electron pressure. Some values of the maximum allowed $|A|$ to have convergence in various cases of k and chosen values of ω_Ω, where $\omega_\Omega = (1+\Gamma_-)\, k^2 v_{s-}^2/\omega_-^2$ and $k_m = \omega_-/v_{s-}$. The values of $|A|$ are estimated using the graphical method elaborated in section 2.3. Clearly including the pressure decreases the convergence and the more the smaller the wavelengths: pressure is relatively more important at short distances than the (long range) electric effects.*

Comments on the convergence

From our results in appendix B.1 we notice that, the coefficients of the higher order terms are polynomial in the parameter

$$\omega_\Omega = (1+\Gamma_-)\frac{k^2 v_{s-}^2}{\omega_-^2} = k^2\,\Gamma_-\,(1+\Gamma_-)\,\Lambda_{D-}^2 = (1+\Gamma_-)\left(\frac{\omega^2}{\omega_-^2} - 1\right),$$

where $\Lambda_{D-}^2 = \varepsilon\, K_-\, n_0^{\Gamma_-}/e^2 n_0^2$. (Note that, if $\Gamma_- = \gamma_-$ then $\Lambda_{D-}^2 = \lambda_{D-}^2$, which is the square of the Debye length). The parameter ω_Ω is a measure of the relative importance of the pressure and the electric action. Moreover the convergence depends now on k, in contrast to the cold plasma case. Substituting equation (2.55) into this parameter, we obtain

$$\omega_\Omega = (\Gamma_- + 1)\left(\frac{k}{k_m}\right)^2. \tag{2.56}$$

To fix the ideas, we take $\Gamma_- = \gamma_- = 5/3$ (monoatomic). For $\omega_\Omega = 0.5$ (i.e. $k/k_m = \sqrt{3}/4$) we find graphically that $|A| < 0.285 \approx 0.775e^{-1}$ for convergence (based on the terms up to fifteenth order). This may be compared with $|A| < e^{-1} \approx 0.367879\cdots$ for cold plasma ($\omega_\Omega = 0$). Other results for the cases $k < k_m$, $k = k_m$ and $k > k_m$ are summarized in table 2.8. Hence *inclusion of the electron motion makes the convergence slower and the domain of convergence smaller.* The more the ω, k curve approaches the oblique asymptote (dispersion relation of sound waves), the worse the convergence as will be explained in section 3.5.

2.4.2 Analysis of perturbation when cosines are used

Here we substitute (2.41) into (2.48) and solve for the required coefficients. With the assistance of *mathematica*, we solved this equation up to order seven (for some results see table 2.9). Comparing these values with those of appendix B.1 we infer the formula

$$b_{-j,j\rho} = \frac{b_{-j\rho}}{2^{j-1}}.$$

It is again possible to infer general formulae for the remaining values as well as for those in velocity and potential. Due to the factor 2^{j-1} in the denominator the convergence is granted up to amplitudes twice as large as in the case of exponentials. In fact exponentials correspond to two waves (or doubling the amplitude). Cf. section 2.3.2.

2.5 Conclusion

In this study we have considered perturbations of ideal unmagnetized plasmas which were represented by systems of nonlinear partial differential equations. Then we reduced each system to a nonlinear differential equation in one unknown, the combined variable $\chi = \omega t + \boldsymbol{k}\cdot\boldsymbol{r}$. The reduction process was done using either the first or the second. These procedures reduce the system respectively to a second and a first order differential equation. The reduced equations were then solved up to higher orders using *mathematica*. Of the two procedures, the first one is the preferred one. This is partially due to the fact that, procedure 1 (indicated in e.g. section 2.4.1.1) produces required results without leading one to tasks such as rejecting some of the results, as it is the case for procedure 2 (given in e.g. section 2.4.1.1). Apparently, the higher the order of the reduced differential equation, the simpler the way to

j	$b_{-j,1\rho}$	$b_{-j,2\rho}$	$b_{-j,3\rho}$	$b_{-j,4\rho}$
1	1			
2	0	$1+\zeta_{2,2}$		
3	0	0	$(9/8)+\zeta_{3,3}$	
4	0	$(1/12)+\zeta_{4,2}$	0	$(4/3)+\zeta_{4,4}$

Table 2.9: *A table for showing some of the coefficients in the particle density when cosines are used in the case of plasma with electron pressure. The ζ's in the table represent the following:* $\zeta_{2,2} = \omega_\Omega/3$, $\zeta_{4,2} = (\omega_\Omega^3/8) + [\omega_\Omega^2\,(110+41\,\Gamma_-)/288] + [\omega_\Omega\,(78+67\,\Gamma_- + 8\,\Gamma_-^2)/288]$, $\zeta_{3,3} = (3\,\omega_\Omega^2/16) + [3\,\omega_\Omega\,(18+\Gamma_-)/64]$, $\zeta_{4,4} = (7\,\omega_\Omega^3/54) + [\omega_\Omega^2\,(54+5\,\Gamma_-)/72] + [\omega_\Omega\,(582+55\,\Gamma_- + 2\,\Gamma_-^2)/360]$.

solve the Fourier analysis. However, as we experience later, the use of further improvements may reverse the order of the most adapted procedures.

In cold and electron plasma (when ions stand still) we noted that the series in exponentials converges with amplitudes less than e^{-1} of the initial density. When pressure terms are involved the convergence is reduced confirming our previous results. E.g. when $k = k_m$, i.e. the intermediate case between cold plasma and sound waves the limiting amplitude for convergence is reduced to 0.13 or 0.35 e^{-1} (see table 2.8). It is hoped that some experimental setup using a strong applied field may verify these results.

Chapter 3

Perturbation Analysis for Unmagnetized Plasma Waves II

3.1 Introduction

In chapter 2 we analyzed the unmagnetized plasma waves. The results there were quite interesting in themselves. Moreover they allowed explicit statements about the convergence of the series. The cases of cold plasma and of plasma with electron pressure were elaborated.

In this chapter we reconsider the system of equations (2.1) - (2.4) and reduce it to a single equation using three methods (in fact two since one of the three is the same as the one given in chapter 2) from which one can determine the higher order coefficients in the expressions of the physical quantities. These two methods are the result of our decision to change the way of tackling the integration of potentials. The procedures now give results which are exactly the same as those determined by the procedure which avoids the integrations. In addition the methods now avoid producing results which leads one into making difficult decisions on which results to reject. Apart from this, they reduced the calculating time substantially. With these reasons, one can justly call these methods "the improved methods", when one compares them with the one considered in the previous chapter. It is therefore our purpose here:

(a) to indicate an improved method, which may shorten the calculating times drastically;

(b) to illustrate the improved method to plasma containing more than one kind of particle species;

(c) to apply the improved method to the electron-positron case;

(d) to compare the computer times required by the procedures (yielding the

same results);
(e) to consider the case where our method gives bad convergence.

The chapter therefore is subdivided as follows: In section 3.2, we then give the three (improved) elimination procedures. The first (which resembles exactly the one given in the preceding chapter) and the second of the improved procedures reduce the system of equations into second and first order nonlinear differential equations respectively, while the third one reduces the system to a fully integrated equation. Again, we have elaborated various alternative procedures because we want to find out which procedure turns out to be the simplest and the fastest in order to calculate the coefficients up to a given order. Perturbation analysis then follows in section 3.3. Here we reconsider the unmagnetized plasma for which the ion particle density is not perturbed (i.e. one component plasma): First cold plasma (i.e. plasma with neither pressure of negative particle nor of positive particles) and next plasma with pressure of negative particles (electrons in this case). We calculate several coefficients to higher orders and again in some cases we obtain analytic expressions and hence confirming the general expressions inferred previously. The calculating times used by the three methods to solve for the coefficients in the case of electron plasma were then compared. This was followed by the analysis of unmagnetized plasma with perturbed electron and ion particle density (i.e. two component plasma) from which higher order coefficients were determined and calculating time for the cold plasma case were compared. In section 3.4 we briefly consider the case of electron-positron plasma. The analysis of sound waves as well as that of ion acoustic waves were then elaborated in sections 3.5 and 3.6 respectively. Finally the conclusion is given in section 3.7.

3.2 Elimination procedures

Here we first let all assumptions and relations given in sections 2.2.3 and 2.2.4 be valid also to this chapter. Then, we applying relations (2.10) to the equations (2.1) - (2.4) to have

$$(\omega + \boldsymbol{k} \cdot \boldsymbol{v}_\alpha) n'_\alpha + n_\alpha \boldsymbol{k} \cdot \boldsymbol{v}'_\alpha = 0, \tag{3.1}$$

$$m_\alpha n_\alpha (\omega + \boldsymbol{v}_\alpha \cdot \boldsymbol{k}) \boldsymbol{v}'_\alpha = -\boldsymbol{k} p'_\alpha - q_\alpha n_\alpha \boldsymbol{k} \varphi', \tag{3.2}$$

$$k^2 \varphi'' = -\frac{1}{\varepsilon} \sum_\alpha (q_\alpha n_\alpha), \tag{3.3}$$

$$p_\alpha = K_\alpha n_\alpha^{\Gamma_\alpha}. \tag{3.4}$$

Integrating (3.1) with respect to χ we obtain

$$(\omega + \boldsymbol{k} \cdot \boldsymbol{v}_\alpha) n_\alpha = \in_\alpha, \tag{3.5}$$

where $\in_\alpha$ is an integration constant associated to the α-th kind of particle species. Using equilibrium quantities we have

$$\in_\alpha = \omega n_{\alpha 0}. \tag{3.6}$$

Then rearranging the equations obtained after substituting (3.6) into (3.1) we have

$$\boldsymbol{k} \cdot \boldsymbol{v}'_\alpha = -\omega n_{\alpha 0} n'_\alpha / n_\alpha^2, \tag{3.7}$$

Multiplying $\boldsymbol{k}$ on both sides of (3.2) followed by the substitution of (3.7) into the obtained equation one gets

$$\frac{m_\alpha \omega^2 n_{\alpha 0}^2 n'_\alpha}{n_\alpha^3} = k^2 \left(\frac{p'_\alpha}{n_\alpha} + q_\alpha \varphi' \right). \tag{3.8}$$

Integrating equation (3.3) with respect to χ we have

$$\varphi' = \frac{-1}{k^2 \varepsilon} \int \sum_\alpha (q_\alpha n_\alpha) \, d\chi, \tag{3.9}$$

where the integration constant may be put equal to zero as required when we substitute the lowest order quantities (i.e. $n_\alpha = n_{\alpha 0}$ (cf. quasi-neutrality) and $\varphi = \varphi_0 = 0$). Then integrating equation (3.9) we get

$$\varphi = \frac{-1}{k^2 \varepsilon} \int \left[\int \sum_\alpha (q_\alpha n_\alpha) \, d\chi \right] d\chi, \tag{3.10}$$

where the integration constant may again be put equal to zero. The equation (3.10) will be used later on in the determination of the coefficients in φ. In fact the main simplification with respect to chapter 2 is the use of these integrations for φ: as n_α is expanded in a series of exponentials (or (co)sines), the integration is straightforward and φ is eliminated much faster. This improved method is particularly useful for the more complicated cases, e.g. when electrons and ions move as otherwise the computation times become very long. Thus using equations (3.4), (3.8) and (3.9) one can complete the reduction process using one of the following procedures; in order to find out which one is the fastest.

3.2.1 Procedures

Procedure 1 (Reduction to nonlinear differential equations of second order in n_α)

Here we differentiate equation (3.8) with respect to χ and then substitute equations (3.3) and (3.4) to have

$$\omega^2\left(n_\alpha n_\alpha'' - 3n_\alpha'^2\right) - \frac{k^2 v_{s\alpha}^2 n_\alpha^{1+\Gamma_\alpha}\left[(\Gamma_\alpha - 2)n_\alpha'^2 + n_\alpha n_\alpha''\right]}{n_{\alpha 0}^{1+\Gamma_\alpha}} + \frac{\omega_\alpha^2 n_\alpha^4}{q_\alpha n_{\alpha 0}^3}\sum_\beta (q_\beta n_\beta) = 0, \tag{3.11}$$

where

$$\omega_\alpha^2 = \frac{q_\alpha^2 n_{\alpha 0}}{\varepsilon m_\alpha} \tag{3.12}$$

and

$$v_{s\alpha}^2 = \frac{K_\alpha \Gamma_\alpha n_{\alpha 0}^{\Gamma_\alpha}}{m_\alpha n_{\alpha 0}} \tag{3.13}$$

are the respective squares of plasma angular frequency and sound velocity for the α-th kind of particle species and $\beta = \alpha$. Using this *second order differential* equation for each α one can easily solve for n_α or rather for the coefficients in the series expression for n_α.

Procedure 2 (Reduction to nonlinear differential equations of first order in n_α)

Putting equations (3.4) and (3.9) into equation (3.8) and rearranging one obtains

$$\left(\omega^2 - \frac{k^2 v_{s\alpha}^2 n_\alpha^{1+\Gamma_\alpha}}{n_{\alpha 0}^{1+\Gamma_\alpha}}\right) n_\alpha' + \frac{\omega_\alpha^2 n_\alpha^3}{q_\alpha n_{\alpha 0}^3}\int \sum_\beta (q_\beta n_\beta)\, d\chi = 0. \tag{3.14}$$

This is a another nonlinear differential equation for each α, however now of *first order*, which can similarly be used, just as (3.11), in the determination of n_α. Note that the integral sign just comes down to a simple algebraic manipulation, corresponding to the essential simplification of the improved method.

Procedure 3 (Reduction to fully integrated equations)

In this case equation (3.8) is integrated with respect to χ yielding, after the substitution of the determined integration constant, the following equation:

$$\frac{\omega^2\left(n_\alpha^2-n_{\alpha 0}^2\right)}{2\,n_\alpha^2}+\frac{k^2\,v_{s\alpha}^2\left(n_{\alpha 0}^{\Gamma_\alpha-1}-n_\alpha^{\Gamma_\alpha-1}\right)}{\left(\Gamma_\alpha-1\right)n_{\alpha 0}^{\Gamma_\alpha-1}}+\frac{\omega_\alpha^2}{q_\alpha\,n_{\alpha 0}}\int\left[\int\sum_\beta\left(q_\beta n_\beta\right)d\chi\right]d\chi=0. \tag{3.15}$$

This *fully integrated* equation for each α may be used equally well as (3.11) and (3.14) in the determination of n_α. Hence having determined the expression for n_α (by using one of the three procedures indicated above), one can then use it in equations (3.4), (3.5) and (3.10) so as to determine the expressions for the pressure, velocity and potential respectively. This shall be illustrated in section 3.2.2.

Comments:
All three nonlinear equations (3.11), (3.14) and (3.15) have to be equivalent (taking into account initial conditions). Equation (3.11) is a differential equation of second order, equation (3.14) is of first order and equation (3.15) is fully integrated. *It is our purpose to show that they lead indeed to the same results and to find out which procedure is the fastest for future use.* In our paper [46] it turned out that the elaboration of the equation of second order is the easiest way. Now the improved method shall give a more subtle outcome. Note however that the improved method does not affect equation (3.11), while it has a strong influence for equation (3.14) and an even stronger one for equation (3.15) where two integrals (which may reduce immediately to a simple algebraic operation) occur.

3.2.2 General equations which can be used in the determination of the coefficients

(I) Particle density
For multi-species plasma, equation (2.26) can be written as

$$n_\alpha=n_{\alpha 0}\sum_{j=0}^{N}a_{\alpha jp}A^j e^{ij\chi}, \tag{3.16}$$

where N goes to infinity. Hence using (3.16) in either equation (3.11), (3.14) or (3.15) yields expressions from which we get the dispersion relation and the values for the coefficients $a_{\alpha jp}$.

(II) Velocity

The velocity terms are investigated by substituting for n_α and v_α in (3.5), where v_α, in agreement with the series for n_α, reads:

$$v_\alpha = \frac{\mathbf{k}}{k} \sum_{j=1}^{N} a_{\alpha j v} A^j e^{ij\chi} \tag{3.17}$$

with $a_{\alpha j v}$ constants to be determined. Hence equation (3.5) becomes

$$\omega = \left(\omega + k \sum_{h=1}^{N} a_{\alpha h v} A^h e^{ih\chi}\right) \left(1 + \sum_{j=1}^{N} a_{\alpha j \rho} A^j e^{ij\chi}\right), \tag{3.18}$$

where $a_{\alpha j \rho}$'s are known at this point. Then from equation (3.18), we determine the values of $a_{\alpha j v}$'s.

(III) Potential

In view of the Fourier analysis for n_α we take

$$\varphi = \sum_{j=1}^{N} a_{j\varphi} A^j \, e^{ij\chi}. \tag{3.19}$$

Plugging this with (3.16) into (3.10) we have

$$\sum_{j=1}^{N} a_{j\varphi} A^j e^{ij\chi} = \sum_{\alpha} \left[\frac{q_\alpha n_{\alpha 0}}{k^2 \varepsilon} \sum_{j=1}^{N} \frac{a_{\alpha j \rho} A^j e^{ij\chi}}{j^2} \right], \tag{3.20}$$

which yields at once the coefficients $a_{j\varphi}$ in the electric potential in terms of the coefficients for the density $a_{\alpha j \rho}$:

$$a_{j\varphi} = \frac{1}{k^2 \, \varepsilon \, j^2} \sum_{\alpha} (a_{\alpha j \rho} \, q_\alpha \, n_{\alpha 0}), \tag{3.21}$$

with $j = 1, 2, \cdots$.

(IV) Pressure

The pressure is given by substituting equation (3.16) (after determining the values of $a_{\alpha j \rho}$'s) into equation (3.4). The convergence is unchanged.

3.3 Perturbation analysis

We have been considering, up to the moment, unmagnetized plasmas of α-th species particles. But for simplicity, as we did previously, we consider here plasma consisting of electrons and one type of ions species, unless otherwise stated. Hence putting (3.16) into equation (3.11), (3.14) and (3.15) with index $\alpha = \pm$, we obtain respectively:

From procedure 1: nonlinear differential equations of second order

$$\omega^2 \left[3 \left(\sum_{l=1}^{N} \sum_{j=1}^{N} l\, j\, a_{\pm l\rho}\, a_{\pm j\rho}\, A^{l+j} e^{i(l+j)x} \right) - \left(\sum_{l=0}^{N} \sum_{j=1}^{N} j^2\, a_{\pm l\rho}\, a_{\pm j\rho}\, A^{l+j}\, e^{i(l+j)x} \right) \right]$$
$$-k^2 v_{s\pm}^2 \left(\sum_{j=0}^{N} a_{\pm j\rho} A^j e^{ijx} \right)^{1+\Gamma_\pm} \left[(2-\Gamma_\pm) \left(\sum_{l=1}^{N} \sum_{j=1}^{N} l\, j\, a_{\pm l\rho}\, a_{\pm j\rho}\, A^{l+j} e^{i(l+j)x} \right) \right.$$
$$\left. - \left(\sum_{l=0}^{N} \sum_{j=1}^{N} j^2\, a_{\pm l\rho}\, a_{\pm j\rho}\, A^{l+j}\, e^{i(l+j)x} \right) \right] \pm \omega_\pm^2 \left(\sum_{j=0}^{N} a_{\pm j\rho} A^j e^{ijx} \right)^4$$
$$\times \left[\sum_{j=1}^{N} A^j e^{ijx} \left(a_{+j\rho} - a_{-j\rho} \right) \right] = 0, \qquad (3.22)$$

from procedure 2: nonlinear differential equations of first order

$$\left[\omega^2 - k^2 v_{s\pm}^2 \left(\sum_{j=0}^{N-1} a_{\pm j\rho} A^j e^{ijx} \right)^{1+\Gamma_\pm} \right] \sum_{j=1}^{N} j\, a_{\pm j\rho}\, A^j\, e^{ijx}$$
$$\pm \omega_\pm^2 \left(\sum_{j=0}^{N} a_{\pm j\rho} A^j e^{ijx} \right)^3 \sum_{j=1}^{N} \frac{A^j e^{ijx} \left(a_{-j\rho} - a_{+j\rho} \right)}{j} = 0 \qquad (3.23)$$

and from procedure 3: fully integrated equations

$$\frac{\omega^2}{2} \left[1 - \left(\sum_{j=0}^{N} a_{\pm j\rho} A^j e^{ijx} \right)^{-2} \right] + \frac{k^2\, v_{s\pm}^2}{(\Gamma_\pm - 1)} \left[1 - \left(\sum_{j=0}^{N} a_{\pm j\rho} A^j e^{ijx} \right)^{\Gamma_\pm - 1} \right]$$
$$\pm \omega_\pm^2 \left[\sum_{j=1}^{N} \frac{A^j e^{ijx}}{j^2} \left(a_{-j\rho} - a_{+j\rho} \right) \right] = 0 \qquad (3.24)$$

with the first coefficient for electrons being absorbed in the amplitude A i.e. $a_{-1\rho} = 1$. We then solve these equations using *mathematica* to determine the dispersion relation and the values for the coefficients. Then we compare the times (duration) required to solve each equation in order to find the most appropriate one.

3.3.1 One component plasma

3.3.1.1 Cold plasma (electrons oscillating while ions stand still)

Here, as in section 2.3, we suppose that the kinetic motion of the charged particles is negligible and thus the temperature can be approximated by zero. As a result, all pressure terms in our formulae have to be zero and hence we put $v_{s+} = 0 = v_{s-}$. In addition we set $\omega_+ = 0$ since here $m_+ >> m_-$.

After applying these modifications to our formulae, we then determined the coefficients $a_{-j\rho}$'s. From each one of the three formulae (i.e. equations (3.22), (3.23) and (3.24)) we recovered all values given in table 2.1 and hence equations (2.28) and (2.29).

The determination of the coefficients a_{-hv}'s and $a_{j\varphi}$'s followed immediately after the determination of $a_{-j\rho}$'s. Here we plugged the obtained $a_{-j\rho}$'s in equations (3.18) and (3.21) and thus recovered the respective results given in tables 2.2 and 2.3. Hence one can again infer the relations given in (2.36) and (2.40) respectively.

With the improved procedures used in this section we were able to confirm all results obtained previously in section 2.3.1.1.

3.3.1.2 Electron plasma waves

Here we no longer neglect the kinetic motion of electrons and thus the pressure of electrons has to be present ($v_{s-} \neq 0$). Thus using one of three procedures, one determines the coefficients in the particle density series. We briefly illustrate this as follows:
We first replace $a_{-j\rho}$ in equations (3.22), (3.23) and (3.24) with $b_{-j\rho}$ so as to avoid confusion between the coefficient determined here with those obtained in section 3.3.1.1. Linearizing these equations one recovers the (Langmuir) dispersion relation given by equation (2.53). We then determine coefficients for the higher order cases. The determined results are the same as those in appendix B.1.1. The times required to determine the first ten coefficients (i.e. $b_{-j\rho}$'s) using the three procedures are given in table 3.1.

For the coefficients in the velocity and potential we again evaluated constants b_{-hv} and $b_{j\varphi}$ (which replaced a_{-jv} and $a_{-j\varphi}$ in equations (3.18) and (3.21) respectively) by using the appropriate *mathematica* programs. The obtained results are the same as those given respectively in appendices B.1.2 and B.1.3, however requiring less calculating time due to the improved method.

The case of electron plasma (ions stand still)			
Coefficients	Time (in seconds) for solving equations for		
	Procedure 1 (Differential equation of second order)	**Procedure 2** (Differential equation of first order)	**Procedure 3** (Fully integrated equation)
$b_{-1\rho}$	-	-	-
$b_{-2\rho}$	0.1	0.1	0.1
$b_{-3\rho}$	0.2	0.2	0.2
$b_{-4\rho}$	0.5	0.4	0.3
$b_{-5\rho}$	1.8	1.0	0.4
$b_{-6\rho}$	5.7	2.5	0.6
$b_{-7\rho}$	16.3	6.3	1.1
$b_{-8\rho}$	46.0	15.4	1.9
$b_{-9\rho}$	123.5	35.8	3.7
$b_{-10\rho}$	275.9	76.8	7.9

Table 3.1: *A table showing the approximate time required in the determination of the coefficients by solving the reduced equation in each of the three elimination procedures. From the table we notice that the time used by procedure 3 in determining the coefficients (in particular those of orders higher than four) is much less than the time spent by the corresponding procedures. In addition, we see that times for the third procedure are nearly doubling when going to next order. For the second procedure they are practically increasing 2.5 times when going to next order. For the first procedure they are nearly tripling with the order. There is no significant difference in times for the lowest coefficients. The times for the first order case are not included simply because the value of $b_{-1\rho}$ is known while that of $b_{+1\rho}$ is determined simultaneously with $\omega^2 = \omega_-^2 + k^2 v_{s-}^2$.*

Comments on the convergence

From table 2.8 (also recovered here), we note that if the incompressibility increases (thus Γ_- increases) ω_Ω increases and thus the convergence decreases. This is physically understandable as the pressure increases relative to the electric forces then the convergence decreases.

Again we suggest that experiments e.g. with a Q-machine should be done with very large perturbations due to an external field in order to verify the reduced convergence.

3.3.2 Two component plasma

We begin this section by replacing the coefficients $a_{\pm j\rho}$ in the equations (3.22), (3.23) and (3.24) by $c_{\pm j\rho}$ and then consider:

3.3.2.1 Cold plasma (both electrons and ions are oscillating)

Here, as it was the case in section 3.3.1.1, we approximate the temperatures by zero and put $v_{s-} \approx 0 \approx v_{s+}$. Plugging this in either (3.22), (3.23) or (3.24) and linearize we get $\omega^2 = 0$ (trivial solution) and dispersion relation for two component cold plasma:

$$\boxed{\omega^2 = \omega_+^2 + \omega_-^2 \equiv \omega_p^2.} \tag{3.25}$$

The dispersion relation (3.25) tells us that both electrons and ions are oscillating around their equilibrium positions with frequency ω_p without propagating. The results here are the same as those obtained when $k \approx 0$.

We then calculated the higher order coefficients. The summary of the determined coefficients is as given in appendix B.2. In table 3.2 we indicates the times used by the respective procedures in the determination of these coefficients simultaneously.

Comment

For the case of electron-positron plasma, where the masses of the charged particles are equal, we put $\omega_+ = \omega_-$ into (3.25) to obtain the dispersion relation for electron-positron plasma: $\omega^2 = 2\omega_-^2$. Then we determine the higher order coefficients by substituting $\omega_+ = \omega_-$ into each of the coefficients given in appendix B.2.

Cold plasma case when both electrons and ions are oscillating			
Coefficients	Time (in seconds) for solving using:		
	Procedure 1 (Differential equation of second order)	Procedure 2 (Differential equation of first order)	Procedure 3 (Fully integrated equation)
$c_{-1\rho}$ and $c_{+1\rho}$	-	-	-
$c_{-2\rho}$ and $c_{+2\rho}$	0.1	0.1	0.1
$c_{-3\rho}$ and $c_{+3\rho}$	0.2	0.2	0.1
$c_{-4\rho}$ and $c_{+4\rho}$	0.3	0.3	0.2
$c_{-5\rho}$ and $c_{+5\rho}$	0.4	0.4	0.2
$c_{-6\rho}$ and $c_{+6\rho}$	0.7	0.6	0.2
$c_{-7\rho}$ and $c_{+7\rho}$	1.1	0.8	0.3
$c_{-8\rho}$ and $c_{+8\rho}$	1.8	1.4	0.4
$c_{-9\rho}$ and $c_{+9\rho}$	2.8	2.0	0.7
$c_{-10\rho}$ and $c_{+10\rho}$	4.4	3.1	1.1
$c_{-11\rho}$ and $c_{+11\rho}$	6.6	4.4	1.9

Table 3.2: *A table for showing the approximate times required for the determination of the coefficients simultaneously using the respective procedures. Again procedure 3 is the fastest. The determination times for the first order case are omitted simply because* $c_{-1\rho}$ *is an assumed constant while* $c_{+1\rho}$ *was determined simultaneously with* $\omega^2 = \omega_-^2 + \omega_+^2$ *in 0.02 seconds.*

3.3.2.2 Two component plasma with pressure

We now include the influence of pressure in our equations and hence use either equation (3.22), (3.23) or (3.24), in the determination of the coefficients in the particle density expressions. As we did previously for the cold plasma case, we first put $c_{-1_\rho} = 1$ in say equation (3.22) and linearize. Then, since there are two unknowns (i.e. ω and c_{+1_ρ}) to be determined, we solve the equations simultaneously and obtain:

$$\omega^2 = \frac{k^2\left(v_{s-}^2 + v_{s+}^2\right) + \omega_-^2 + \omega_+^2 \pm \sqrt{4\,\omega_-^2\,\omega_+^2 + [k^2\,(v_{s-}^2 - v_{s+}^2) + \omega_-^2 - \omega_+^2]^2}}{2} \tag{3.26}$$

and c_{+1_ρ} which is given in appendix B.3. Equation (3.26) is the solution to the dispersion relation:

$$\frac{\omega_-^2}{\omega^2 - k^2 v_{s-}^2} + \frac{\omega_+^2}{\omega^2 - k^2 v_{s+}^2} = 1, \tag{3.27}$$

while c_{+1_ρ} is the coefficient of the first order term in the expression of the ion particle density. Apart from the above equations, we also determined other higher order coefficients (up to order five) in both series of electrons and of ions. However, we give in appendix B.3 only the expressions for the coefficients of the first and the second order terms, since the other ones are very long. The expressions for the velocity and potential were determined by using (3.18) and (3.21) respectively.

Comments on the dispersion relations

1. Cold plasma or long waves
   If $kv_{s\pm} << \omega$, i.e. $k << k_m$, hence for very long waves, we recover from (3.27) dispersion relation (3.25). "Long" depends on the ratio of $\omega/v_{s\pm}$ i.e. the ratio of electric and pressure forces. If $T_\pm \approx 0$, hence $v_{s\pm} \approx 0$, then virtually all wavelengths are allowed.

   For the case where $\omega^2 << k^2 v_{s\pm}^2$ or when $\omega^2 = 0$, the dispersion relation (3.27) becomes $k^2 = -(1/\Lambda_{D-} + 1/\Lambda_{D+})$, where $\Lambda_{D\pm} = \Gamma_\pm \omega_\pm^2 / v_{s\pm}^2$. In fact this relation shows that $k^2 \rightarrow \infty$ is prohibited and that there is a limit for k. In fact too small wavelengths (below the Debye length) are not compatible with the neutrality condition.

2. Ions stand still
   Substituting $\omega_+ = 0 = v_{s+}$ into (3.27) one recovers the dispersion relation (2.53) obtained when ions are not moving.

3. Dust plasma and bucky balls
   Similarly putting $\omega_- = 0 = v_{s-}$ into (3.27) yields the dispersion relation for the case when the negative particles stand still:

$$\omega^2 = \omega_+^2 + k^2 v_{s+}^2. \tag{3.28}$$

   This is the case when the mass of the negative particles is much higher than the ion mass. This was long considered as an unphysical situation. However in the tails of comets and in the interplanetary space e.g. the dust particles are sometimes negatively charged due to electrons which are attached to them. Those dust particles may easily have masses of 10^8 amu and accumulate hundreds or thousands of electrons [41, 42, 43, 44].

   In the laboratory (and in astrophysical phenomena) this situation may e.g. occur in the experiments with bucky balls or fullerenes, i.e. carbon sixty (or seventy, ...) which are constituted by 60 carbon atoms. Hence their mass is 720 amu (respectively 840 amu for 70 carbon atoms). The fullerenes have the property that they may attach an electron so that they become negatively charged. The frequencies are of course much lower than for an "ordinary plasma" [45]. However, the situation is usually much more complicated because in many cases a plasma with three or more components occurs, as not all electrons are attached to the heavy particles. One has then a dispersion relation of the type (3.27) with more terms.

3.4 Electron-positron plasma

Photons with energy higher than 1 MeV may disintegrate in an electron and a positron (rest energy 511 keV). Around pulsars one may have a plasma of extremely high temperature (over 10^{10}K), which allows the photons to disintegrate. As the density of photons is extremely high the density of electrons and positrons is extremely high as well (the density may be in the range of 10^{32}m^{-3} or even 10^{34}m^{-3} (cf. note below)) [47, 48].

In electron-positron plasma we have $m_+ = m_-$ and thus $\omega_+ = \omega_-$. Using this and assuming that $T_+ = T_-$ or $v_{s+} = v_{s-}$, we get from equation (3.27) the dispersion relation:

$$\omega^4 - 2\omega^2\left(\omega_-^2 + k^2 v_{s-}^2\right) + 2\omega_-^2 k^2 v_{s-}^2 + k^4 v_{s-}^4 = 0 \tag{3.29}$$

which is a biquadratic equation whose solutions are

$$\omega^2 = 2\omega_-^2 + k^2 v_{s-}^2 \quad \text{and} \quad \omega^2 = k^2 v_{s-}^2. \tag{3.30}$$

These modes are respectively the Langmuir and sound waves. We then calculate higher order coefficients

Graphical representation

We sketch the graphs of $\omega^2 = 2\omega_-^2$ and $\omega^2 = 2\omega_-^2 + k^2\, v_{s-}^2$ on the same graph (see figure 2.4). From that figure we notice that $\omega^2 = 2\omega_-^2$ meets the graph of $\omega^2 = \omega_-^2 + k^2\, v_{s-}^2$ (which was considered previously in section 2.4) at the point $(\omega_m,\, k_m)$, where k_m is given by (2.55) and ω_m (or $\omega_m = \sqrt{2}\,\omega_-$) is the the value of the frequency when $k = k_m = \left(\Lambda_{D-}\sqrt{\Gamma_-}\right)^{-1}$. It is interesting to see the two graphs cross each other at point. This gives a parameter k_m a much stronger basis as it can be used for comparison purposes.

Maybe the nonlinear effects and the convergence can be verified by observing strong signals from pulsars passing through their electron-positron plasma atmosphere.

Higher order coefficients

In table 3.3 we give the times required to determine the coefficients in the perturbed particle density for the case of electron-positron plasma when $\omega^2 = 2\omega_-^2 + k^2 v_{s-}$ is used in (3.22), (3.23) or (3.24) with $a_{\pm j\rho}$ being replaced by $cc_{\pm j\rho}$ and $cc_{-1\rho} = 1$ (using cc as a notation). Using the other dispersion relation (i.e. $\omega^2 = k^2 v_{s-}^2$) instead, one obtains $cc_{\pm 2\rho}$ to be infinite as it should be because this corresponds to pure sound waves.

3.5 Sound waves: An example of the case where the method gives bad convergence

Our method worked very well with the cases of plasmas considered in the preceding sections. However, the method gives no convergence when one applies it to sound waves. Though simple, sound waves are baffling due to zero convergence (When including effects like viscosity the convergence depends on k). Nevertheless, the correct dispersion relation is obtained in first order. We illustrate these factors in the analysis below.

Neglecting viscosity, one can describe sound waves as:

$$\partial_t \rho + div(\rho v) = 0, \tag{3.31}$$

$$\rho \frac{dv}{dt} = -\nabla p, \tag{3.32}$$

Electron-positron case			
Coefficients	Time (in seconds) for solving by		
	Procedure 1 (Second order differential equation)	Procedure 2 (First order differential equation)	Procedure 3 (Fully integrated equation)
ω & $cc_{+1\rho}$	0.062	0.094	0.125
$cc_{\pm 2\rho}$	0.062	0.094	0.125
$cc_{\pm 3\rho}$	0.344	0.25	0.25
$cc_{\pm 4\rho}$	1.031	0.688	0.469
$cc_{\pm 5\rho}$	3.5	1.797	0.687
$cc_{\pm 6\rho}$	11.172	4.812	1.172
$cc_{\pm 7\rho}$	31.75	11.875	1.984
$cc_{\pm 8\rho}$	78.391	28.485	3.719
$cc_{\pm 9\rho}$	174.937	60.39	7.391
$cc_{\pm 10\rho}$	364.172	126.516	15.891

Table 3.3: *A table for showing the approximate times used by the appropriate procedures to determine the respective coefficients simultaneously. As in tables 3.1 and 3.2, procedure 3 (fully integrated equation) is the fastest. When going to a higher order the calculating time increases roughly by a factor 2 for the third procedure, by a factor 2.5 for the second one and by about 3 for the first one.*

$$p = K\rho^{\Gamma}, \tag{3.33}$$

where ρ is the density of the gas. As in section 2.3, all quantities are taken here to be functions of χ alone. Thus, the above system can be reduced to

$$\left(\omega^2\rho_0^2 - k^2\Gamma K\rho^{\Gamma+1}\right)\rho' = 0 \tag{3.34}$$

which implies that either

$$\rho' = 0 (\Rightarrow \rho \text{ is a constant}) \tag{3.35}$$

or

$$\omega^2\rho_0^2 = k^2\Gamma K\rho^{\Gamma+1}. \tag{3.36}$$

Expression (3.35) would require that our system is incompressible, prohibiting sound waves (i.e. requiring the amplitude to be vanishingly small). Equation (3.36) (which yields the dispersion relation for sound waves when zero order quantities are substituted) requires ρ to be constant too, unless $\Gamma = -1$. The case $\Gamma = -1$ has no physical sense but when we infer it in the equations (3.31) - (3.33) it turns out that ρ has to be again constant (Moreover, when $\Gamma < 0$, equation (3.36) would require $\omega^2 < 0$, meaning instability). Clearly in all cases the radius of convergence is zero. For more insight we analyze equations (3.31) - (3.33) after substituting various orders.

3.5.1 Attempt with series development

In this part we consider again equations (3.31) - (3.33) and analyze them after substituting quantities from various orders.

Zero order
From (3.31) - (3.33) it follows that ρ_0 and p_0 are constant with respect to space and time.

First order
Put $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$ and $\boldsymbol{v} = \boldsymbol{v}_1$ into the system (3.31) - (3.33) and simplify to get

$$\partial_t\rho_1 = -\rho_0\nabla\cdot\boldsymbol{v}_1, \tag{3.37}$$

$$\rho_0\partial_t\boldsymbol{v}_1 = -v_s^2\nabla\rho_1. \tag{3.38}$$

Taking divergence on (3.38) we have

$$\rho_0\partial_t(\nabla\cdot\boldsymbol{v}_1) = -v_s^2\Delta\rho_1, \tag{3.39}$$

since ∂_t and ∇ can commute. Substituting (3.37) into (3.39) we get

$$(\partial_{tt}^2 - v_s^2\Delta)\rho_1 = 0. \tag{3.40}$$

We then let

$$\rho = \rho_0 + \rho_1 + \rho_2 + \cdots = \sum_j \rho_0\, \alpha_{js}\, A^j\, e^{ij\chi}, \tag{3.41}$$

where $\alpha_{0s} = \alpha_{1s} = 1$. Substituting this into (3.40) leads to the dispersion relation for sound waves:

$$\boxed{\omega^2 = k^2 \Gamma K \rho_0^{\Gamma-1} \equiv k^2 v_s^2.} \tag{3.42}$$

For p_1 and v_1 we find

$$p_1 = v_s^2 \rho_1 = v_s^2 A \rho_0 e^{i\chi} \tag{3.43}$$

and

$$\boldsymbol{v}_1 = -\frac{\boldsymbol{k}}{k}\frac{\omega}{k} A e^{i\chi} = -\frac{\boldsymbol{k} v_s^2 \rho_1}{\omega \rho_0}. \tag{3.44}$$

Hence no problem up to the first order.

<u>Second order</u>

Expanding and neglecting all quantities involving orders higher than two, our system becomes

$$\partial_t \rho_2 + \rho_0 \nabla \cdot \boldsymbol{v}_2 + \rho_1 \nabla \cdot \boldsymbol{v}_1 + (\boldsymbol{v}_1 \cdot \nabla)\, \rho_1 = 0, \tag{3.45}$$

$$\rho_0 \left[\partial_t \boldsymbol{v}_2 + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_1\right] + \rho_1 \partial_t \boldsymbol{v}_1 = -v_s^2 \nabla \left[\rho_2 + \frac{(\Gamma - 1)\rho_1^2}{2\rho_0}\right]. \tag{3.46}$$

Eliminating $\boldsymbol{v}_2$ from these yields

$$(\partial_{tt}^2 - v_s^2 \Delta)\rho_2 = -[(\partial_t \rho_1)(\nabla \cdot \boldsymbol{v}_1) + \rho_1 \partial_t (\nabla \cdot \boldsymbol{v}_1) + (\partial_t \boldsymbol{v}_1)(\nabla \rho_1) + \boldsymbol{v}_1 \cdot \nabla(\partial_t \rho_1)] +$$
$$\rho_0 \nabla(\boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_1) + (\nabla \rho_1)\partial_t \boldsymbol{v}_1 + \rho_1 \nabla \partial_t \boldsymbol{v}_1 + v_s^2 \Delta \left[\frac{(\Gamma - 1)\rho_1^2}{2\rho_0}\right]. \tag{3.47}$$

Putting (3.44) into (3.47) and replacing ρ_2 with $\alpha_{2s} \frac{\rho_1^2}{\rho_0}$ yields

$$\alpha_{2s}\, \rho_1^2 (\omega^2 - k^2 v_s^2) = \rho_1^2 [\omega^2 + k^2 v_s^2 \frac{(\Gamma - 1)}{2}].$$

Since $\omega^2 - v_s^2 k^2$ is zero we must have $\Gamma = -1$, which is physically meaningless or α_{2s} has to be infinite, which just means that the series diverges for any amplitude.

From the analysis above it is clear that, for pure sound waves (no electromagnetic interaction) the series either diverges (meaning that instability has to occur or one of the assumptions is wrong) or gives results which are physically meaningless. However sound waves do exist. This motivates us to investigate further the sound wave problem in question. We therefore generalize the above approach so as to have more insight about our method and the problem itself.

3.5.2 A more general approach

From (3.31) - (3.33) we have

$$\partial_t \rho + div(\rho \boldsymbol{v}) = 0 \tag{3.48}$$

and

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -K\Gamma\rho^{\Gamma-2}\boldsymbol{\nabla}\rho. \tag{3.49}$$

Let $\rho = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \epsilon^3\rho_3 + \cdots$ and $\boldsymbol{v} = \epsilon\boldsymbol{v}_1 + \epsilon^2\boldsymbol{v}_2 + \epsilon^3\boldsymbol{v}_3 + \cdots$, where ϵ is now a small arbitrary constant.

<u>First order</u>
For the first order we have

$$\partial_t \rho_1 + div(\rho_0 \boldsymbol{v}_1) = 0 \tag{3.50}$$

and

$$\partial_t \boldsymbol{v}_1 = -K\Gamma\rho_0^{\Gamma-2}\boldsymbol{\nabla}\rho_1. \tag{3.51}$$

Substituting (3.50) into the equation resulted from taking divergence on (3.51), we obtain a wave equation given in (3.40). This has a general solution which can be written as the sum of eight terms of the type

$$\rho_1 = f(\chi),$$

with $\omega/k = v_s$ by definition and f an arbitrary function. Substituting this into (3.50) and (3.51) yields

$$\omega f' + div\rho_0 \boldsymbol{v}_1 = 0 \tag{3.52}$$

and

$$\partial_t \boldsymbol{v}_1 = -\frac{v_s^2 \boldsymbol{k}}{\rho_0} f'. \tag{3.53}$$

Integrating (3.53) with respect to t we get

$$\boldsymbol{v}_1 = -\frac{v_s^2 \boldsymbol{k}}{\omega\rho_0} f + C(\boldsymbol{r}). \tag{3.54}$$

We may put $C(\boldsymbol{r}) = \boldsymbol{0}$ since for vanishing f we must have vanishing $\boldsymbol{v}_1$.

<u>Second order</u>
Here we see that

$$\partial_t(\rho_1 + \epsilon\,\rho_2) + \rho_0\,\boldsymbol{\nabla}\,(\boldsymbol{v}_1 + \epsilon\,\boldsymbol{v}_2) + \epsilon\,(\rho_1\,\boldsymbol{\nabla}\,\boldsymbol{v}_1 + \boldsymbol{v}_1\,\boldsymbol{\nabla}\,\rho_1) = 0$$

and

$$\partial_t \boldsymbol{v}_1 + \epsilon\,(\partial_t \boldsymbol{v}_2 + \boldsymbol{v}_1 \cdot \nabla\, \boldsymbol{v}_1) = -\frac{v_s^2}{\rho_0}\left[\nabla(\rho_1 + \epsilon\,\rho_2) + \frac{\epsilon\,(\Gamma-2)}{\rho_0}\,\rho_1\,\nabla\,\rho_1\right].$$

Substituting respectively (3.50) and (3.51) into these we get

$$\partial_t \rho_2 + \rho_0 \nabla \boldsymbol{v}_2 + \rho_1 \nabla \boldsymbol{v}_1 + \boldsymbol{v}_1 \nabla \rho_1 = 0 \tag{3.55}$$

and

$$\partial_t \boldsymbol{v}_2 + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_1 = -\frac{v_s^2}{\rho_0}\left[\nabla \rho_2 + \frac{(\Gamma-2)}{\rho_0}\rho_1 \nabla \rho_1\right]. \tag{3.56}$$

Take divergence on (3.56) and then substitute (3.55) into the new expression in order to get rid of $\boldsymbol{v}_2$ yielding

$$(\partial_{tt}^2 - v_s^2 \Delta)\rho_2 = \frac{v_s^2(\Gamma-2)}{\rho_0}(\nabla \rho_1 \cdot \nabla \rho_1 + \rho_1 \Delta \rho_1) + \rho_0(\nabla \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_1 + \boldsymbol{v}_1 \Delta \boldsymbol{v}_1) - \partial_t(\rho_1 \nabla \boldsymbol{v}_1 + \boldsymbol{v}_1 \nabla \rho_1). \tag{3.57}$$

For illustration purposes, we consider the following two cases

(a) Let $f = e^{i\chi}$.

Putting this into (3.54) and then combining the obtained equation with (3.57) to have

$$\left(\partial_{tt}^2 - v_s^2 \Delta\right)\rho_2 = C_h\, f^2$$

where $C_h = -[2k^2 v_s^2(\Gamma+1)]/(\rho_0)$. In a general way the solution of the differential equation of this type (i.e. with differentials on the left hand side and a given function on the right hand side) consists of the sum of two parts: one part is a solution of the equation without the right hand side (omitting the given function), the other part is a particular solution with the right hand side (note: any particular solution will do!). The general solution of the left hand side is (restricting $\boldsymbol{k} \cdot \boldsymbol{r}$ to kx)

$$f_a(\omega\, t + k\, x) + f_b(\omega\, t - k\, x)$$

with f_a and f_b arbitrary functions of their argument. However the solution without right hand side has nothing to do with the first order solution under consideration, so we drop it. For a particular solution with right hand side one has usually to take a function like the one on the right hand side. However here the f on the right hand side satisfies the left hand side (this is called "absorption") and one has to adapt the function with a supplementary factor. Hence we find that

$$\rho_2 = \frac{C_h\, t\, e^{2\,i\,(\omega\, t + k\, x)}}{4\, i\, \omega}$$

is a particular solution (a similar solution with x in front of the exponential is a particular solution too: however, the difference between both particular solutions belongs to the solutions of the left hand side.) This is the relevant solution for our nonlinear analysis. However it is no more periodic. We may construct the higher order terms (involving higher powers of t or x), but clearly the solution will have limited application: for large t the oscillations become very wild. To avoid this one often uses an adapted ω' which is shifted somewhat from ω, for the second order term. However, it is clear that this again means that the solution is not really periodic although the first order term is periodic. In either of those procedures the convergence or the range of application is not clear, cf. the introduction.
(b) Let $f = (\omega\, t + k\, x)^j$, where $j = 1, 2, \cdots$. Using this in (3.57) gives

$$\left(\partial^2_{tt} - v_s^2 \Delta\right) \rho_2 = C_q\, f^{2(1+1/j)}.$$

where $C_q = k^2 v_s^2 (\Gamma + 1) j(2j - 1)/\rho_0$. In a similar way as for case (a), we find the particular solution:

$$\rho_2 = \frac{C_q\, t\, (\omega\, t + k\, x)^{3+2/j}}{2\,(3\, +\, 2/j)\,\omega}$$

The solution is no more dependent on χ alone but separately on t as well. Again in higher order terms powers of t (or x) may occur. However the situation is quite different from the above: indeed here the first order solution was not a periodic function, rather a kind of traveling wave. However the nonlinear terms alter this character.

In our analysis of sound waves the phase velocity is constant ($= v_s$) and the dispersion relation gives a straight line through the origin. Hence $\omega/k = n\,\omega/n\,k$ for all (ω, k), n arbitrary. This makes $(\partial^2_{tt} - v_s^2 \Delta) = 0$ for all (ω, k) and all n. However the right hand side differs from case to case depending on the choice of f, but is different from zero and thus leads to infinite coefficients in order to compensate for the vanishing of the left hand side.

Our method proved to be inadequate for sound waves analysis due to zero convergence. However, it works with plasma. Let us now apply our method to ion acoustic waves, a phenomenon which is analogous to sound waves. The two phenomena are similar simply because both are pressure waves propagating from one layer to the next though using different mechanisms.

3.6 Ion acoustic waves

Chen [32, p. 95] pointed out that in the absence of collisions, ordinary sound waves would not occur. But in plasmas, ions can still transmit vibrations to each other because of their charge, and acoustic waves can occur through the intermediary of an electric field. Let us now use our method in analyzing ion acoustic waves to higher orders.

3.6.1 Analysis with plasma approximation

In this analysis we first adjust our basic system of equations by the so-called *plasma approximation*. According to Chen [32, p. 77], the *plasma approximation* is a mathematical shortcut whereby it is possible to assume (in plasma) that $n_- = n_+$ and $\nabla \cdot \boldsymbol{E} \neq 0$ at the same time. It (*plasma approximation*) is a fundamental trait of plasmas, one which is difficult for the novice to understand, Chen states: "*Do not use Poisson's equation to obtain* $\boldsymbol{E}$ *unless it is unavoidable!*". We may therefore replace the Poisson equation (3.3) by $n_- = n_+$. Apart from this, since we are now dealing with ion acoustic waves, one neglects electron inertia because the ion plasma frequency (ω_+) is at least by factor of $(m_-/m_+)^{1/2} = 43$ smaller than ω_- (for hydrogen). At such low frequencies the electrons react almost without any inertia to the change in the electric field [49, pg.203]. Hence three approximations are involved here:

- Plasma approximation: The Poisson equation is replaced by $n_- = n_+$
- The inertia term in the equation of motion of the electrons is neglected, although the higher variations compensate to some extent for the small electron mass
- Each species feels its own pressure only.

Under these conditions equations (3.1) - (3.4), with α being put equal to $\pm$, become:

$$(\omega + \boldsymbol{k} \cdot \boldsymbol{v}_+)n'_+ + n_+\boldsymbol{k} \cdot \boldsymbol{v}'_+ = 0, \tag{3.58}$$

$$m_+n_+ (\omega + \boldsymbol{v}_+ \cdot \boldsymbol{k})\, \boldsymbol{v}'_+ = -\boldsymbol{k}p'_+ - en_+\boldsymbol{k}\varphi', \tag{3.59}$$

$$-\boldsymbol{k}p'_- + en_-\boldsymbol{k}\varphi' = 0, \tag{3.60}$$

$$n_- = n_+ = n, \tag{3.61}$$

$$p_\pm = K_\pm n_\pm^{\Gamma_\pm}. \tag{3.62}$$

Putting (3.60) and (3.61) into (3.59) we have

$$m_+n\, (\omega + \boldsymbol{v}_+ \cdot \boldsymbol{k})\, \boldsymbol{v}'_+ = -\boldsymbol{k}\left(p'_+ + p'_-\right). \tag{3.63}$$

Multiply by $\boldsymbol{k}$ on both sides of (3.63), after substituting (3.62) into it, to get

$$m_+n\, (\omega + \boldsymbol{v}_+ \cdot \boldsymbol{k})\, \boldsymbol{k} \cdot \boldsymbol{v}'_+ = -k^2[K_+\Gamma_+n^{\Gamma_+-1} + K_-\Gamma_-n^{\Gamma_--1}]n'. \tag{3.64}$$

Integrate (3.58) with respect to χ to obtain

$$(\omega + \boldsymbol{k} \cdot \boldsymbol{v}_+)n = \epsilon_+ = \omega n_0. \tag{3.65}$$

Substitute this into (3.64) and then eliminate $\boldsymbol{k} \cdot \boldsymbol{v}'_+$ by substituting (3.58) into the obtained equation to have

$$\boxed{m_+ \omega^2 n_0^2 = k^2(K_+\Gamma_+ n^{\Gamma_+ +1} + K_-\Gamma_- n^{\Gamma_- +1}).} \tag{3.66}$$

Zero order

We substitute $n = n_0$ into (3.66) to get

$$\boxed{\omega^2 = \frac{k^2(K_+\Gamma_+ n_0^{\Gamma_+ -1} + K_-\Gamma_- n_0^{\Gamma_- -1})}{m_+} = k^2\left(v_{s+}^2 + \frac{K_-\Gamma_- n_0^{\Gamma_- -1}}{m_+}\right).} \tag{3.67}$$

This is the dispersion relation for ion acoustic waves. If we let $\Gamma_\pm = \gamma_\pm$, the respective ratios of the specific heats of ions and electrons, then (3.67) may be written

$$\omega^2 = \frac{k^2(\gamma_+ p_{+0} + \gamma_- p_{-0})}{m_+} = \frac{k^2(\gamma_+ k_B T_+ + \gamma_- k_B T_-)}{m_+},$$

where k_B is the Boltzmann's constant and $T_\pm$ are the temperatures of ions and electrons respectively. Dividing on both sides of this equation by k^2, we have the phase velocity of ion acoustic waves

$$\boxed{\frac{\omega}{k} = \left(\frac{\gamma_+ k_B T_+ + \gamma_- k_B T_-}{m_+}\right)^{1/2} \equiv v_{sp},} \tag{3.68}$$

where v_{sp} is the sound speed in plasma [32] or ion acoustic speed [49].

For higher orders we again have zero convergence, just as it was the case in section 3.5. As it is explained there this is clear from the dispersion relation (3.68), with ω linearly proportional to k.

3.6.2 Analysis using the Poisson equation

Here we keep all assumptions considered in section 3.6.1 but replace equation (3.61) with the Poisson equation:

$$k^2\varphi'' = \frac{e}{\varepsilon}(n_- - n_+). \tag{3.69}$$

Then our reduction process begins by substituting (3.58), (3.62) and (3.65) into the equation obtained after multiplying on both sides of (3.59) by $\boldsymbol{k}$. This gives

$$m_+ \frac{\omega^2 n_0^2 n'_+}{n_+^2} = k^2 (K_+ \Gamma_+ n_+^{\Gamma_+ - 1} n'_+ + e n_+ \varphi'). \tag{3.70}$$

Substituting (3.62) into (3.60) we have

$$\varphi' = \frac{K_- \Gamma_-}{e} n_-^{\Gamma_- - 2} n'_-, \tag{3.71}$$

which is integrated with respect to χ to give

$$\varphi = \frac{K_- \Gamma_-}{e(\Gamma_- - 1)} n_-^{\Gamma_- - 1} + C_{ss},$$

where C_{ss} is an integration constant. Putting zero order quantities into this equation we obtain

$$C_{ss} = -\frac{K_- \Gamma_-}{e(\Gamma_- - 1)} n_0^{\Gamma_- - 1},$$

hence φ becomes

$$\varphi = \frac{K_- \Gamma_-}{e(\Gamma_- - 1)} \left[n_-^{\Gamma_- - 1} - n_0^{\Gamma_- - 1} \right]. \tag{3.72}$$

Then substituting (3.72) into (3.69) and rearranging yields

$$n_+ = \left(\frac{e(\Gamma_- - 1)}{K_- \Gamma_-} \varphi + n_0^{\Gamma_- - 1} \right)^{\frac{1}{\Gamma_- - 1}} - \frac{\varepsilon k^2 \varphi''}{e}, \tag{3.73}$$

which combines with (3.70) to give

$$\begin{aligned} m_+ \omega^2 n_0^2 & \left[\left(\frac{e(\Gamma_- - 1)}{K_- \Gamma_-} \varphi + n_0^{\Gamma_- - 1} \right)^{\frac{1}{\Gamma_- - 1}} - \frac{\varepsilon k^2 \varphi''}{e} \right]' \\ = k^2 & \left\{ K_+ \Gamma_+ \left[\left(\frac{e(\Gamma_- - 1)}{K_- \Gamma_-} \varphi + n_0^{\Gamma_- - 1} \right)^{\frac{1}{\Gamma_- - 1}} - \frac{\varepsilon k^2 \varphi''}{e} \right]^{\Gamma_+ + 1} \right. \\ & \times \left[\left(\frac{e(\Gamma_- - 1)}{K_- \Gamma_-} \varphi + n_0^{\Gamma_- - 1} \right)^{\frac{1}{\Gamma_- - 1}} - \frac{\varepsilon k^2 \varphi''}{e} \right]' \\ & \left. + e \left[\left(\frac{e(\Gamma_- - 1)}{K_- \Gamma_-} \varphi + n_0^{\Gamma_- - 1} \right)^{\frac{1}{\Gamma_- - 1}} - \frac{\varepsilon k^2 \varphi''}{e} \right]^3 \varphi' \right\}. \end{aligned} \tag{3.74}$$

We use this equation in solving for φ.

First order

Linearizing (3.74) we get

$$\frac{m_+\,\omega^2}{k^2}\left[\frac{en_0^{2-\Gamma_-}\varphi_1'}{K_-\Gamma_-}-\frac{\varepsilon k^2\varphi_1'''}{e}\right]=K_+\Gamma_+n_0^{\Gamma_+-1}\left[\frac{en_0^{2-\Gamma_-}\varphi_1'}{K_-\Gamma_-}-\frac{\varepsilon k^2\varphi_1'''}{e}\right]+en_0\varphi_1'.$$

Putting $\varphi_1 = \xi\ e^{i\chi}$ (ξ is a constant amplitude) into this and rearranging yields the following general dispersion relation

$$\boxed{\frac{\omega^2}{k^2}=v_{s+}^2+\frac{1}{m_+}\left[\frac{e^2K_-\Gamma_-n_0^{\Gamma_-+1}}{e^2n_0^2+\varepsilon k^2K_-\Gamma_-n_0^{1-\Gamma_-}}\right].}\tag{3.75}$$

If we then put (3.75), with $\Gamma_\pm=\gamma_\pm$, into the form

$$\frac{\omega^2}{k^2}=\frac{\gamma_+k_BT_+}{m_+}+\frac{1}{m_+}\left[\frac{e^2\gamma_-n_0^2k_BT_-}{e^2n_0^2+\varepsilon k^2\gamma_-n_0k_BT_-}\right],$$

we obtain the dispersion relation [32]:

$$\boxed{\frac{\omega^2}{k^2}=\frac{\gamma_+k_BT_+}{m_+}+\frac{k_BT_-}{m_+}\left[\frac{\gamma_-}{1+k^2\gamma_-\lambda_D^2}\right],}\tag{3.76}$$

where $\lambda_D^2=\frac{\varepsilon k_BT_-}{n_0e^2}$ is the square of the Debye length.

Second order

Now ω and k are no longer purely proportional and the higher order analysis makes sense. Hence putting $\varphi=\varphi_1+\varphi_2$ into (3.74) and then expanding it neglecting quantities of order three and above we get

$$\begin{aligned}
&m_+\omega^2\,n_0^2\left[\frac{en_0^{(2-\Gamma_-)}(\varphi_1'+\varphi_2')}{K_-\Gamma_-}+\frac{e^2n_0^{3-2\Gamma_-}(2-\Gamma_-)\varphi_1\varphi_1'}{2K_-^2\Gamma_-^2}-\frac{\varepsilon k^2(\varphi_1'''+\varphi_2''')}{e}\right]\\
&=k^2\left\{K_+\Gamma_+\left[\left(n_0^{\Gamma_++1}+\frac{en_0^{\Gamma_+-\Gamma_-+2}(\Gamma_++1)\varphi_1}{K_-\Gamma_-}\right)-(\Gamma_++1)n_0^{\Gamma_+}\frac{\varepsilon k^2\varphi_1''}{e}\right]\right.\\
&\times\left[\frac{en_0^{(2-\Gamma_-)}\varphi_1'}{K_-\Gamma_-}-\frac{\varepsilon k^2\varphi_1'''}{e}\right]+K_+\Gamma_+n_0^{\Gamma_++1}\left[\frac{en_0^{(2-\Gamma_-)}\varphi_2'}{K_-\Gamma_-}+\frac{e^2n_0^{3-2\Gamma_-}(2-\Gamma_-)\varphi_1\varphi_1'}{2K_-^2\Gamma_-^2}\right.\\
&\left.\left.-\frac{\varepsilon k^2\varphi_2'''}{e}\right]+e\left[n_0^3\varphi_1'+\frac{3en_0^{3-\Gamma_-+1}\varphi_1\varphi_1'}{K_-\Gamma_-}-3n_0^2\frac{\varepsilon k^2\varphi_1''\varphi_1'}{e}+n_0^3\varphi_2'\right]\right\}.
\end{aligned}\tag{3.77}$$

Substituting $\varphi_2=c_2n_0A^2e^{2i\chi}=c_2\varphi_1^2/n_0$ into this yields

$$c_2 = \left\{\frac{2e}{\omega_+^2}\left(\frac{\omega^2}{k^2} - v_{s+}^2\right)\left(\frac{1}{\Lambda_D^2\Gamma_-} + 4k^2\right) - 2e\right\}^{-1}\left\{-\frac{\omega^2 \varepsilon n_0 m_+(2-\Gamma_-)}{2k^2 K_- n_0^{\Gamma_-}\Gamma_-^2\Lambda_D^2}\right.$$

$$\left. + \frac{\varepsilon v_{s+}^2}{\omega_+^2}\left[(\Gamma_+ + 1)\left(\frac{1}{\Lambda_D^2\Gamma_-} + k^2\right)^2 + \frac{(2-\Gamma_-)}{2\Lambda_D^4\Gamma_-^2}\right] + 3\varepsilon\left(\frac{1}{\Lambda_D^2\Gamma_-} + k^2\right)\right\},$$

where $\Lambda_D = \frac{\varepsilon K_- n_0^{\Gamma_-}}{e^2 n_0^2}$.

Hence without plasma approximation (i.e putting $n_+ = n_-$, but still neglecting the electron inertia) we obtained however, a generalized dispersion relation (generalized because the constant Γ, instead of the usual ratio of specific heats (γ) was used). Apart for the dispersion relation, we obtained coefficients up to order two. Other higher order coefficients can also be generated but they are very long. The convergence can then be studied but one needs more time and if possible, more powerful computer.

3.7 Conclusion

As we did previously in chapter 2, we have considered in this chapter the perturbations of ideal unmagnetized plasma which was represented by a system of nonlinear partial differential equations. The system was then reduced to a nonlinear equation of one unknown. The reduction process was completed using either one of three improved procedures. These procedures, as it is in chapter 2, reduce the system respectively to a second and a first order ordinary differential equation and a fully integrated equation. An improved method as compared to our paper [46] allows a drastic reduction in calculating times. Moreover the improved method avoids the difficult choice in rejecting some solutions which troubled procedure 2 in the previous work. It is very satisfying that all three procedures lead to the same results. However, the required calculating time may be strongly different.

The fully integrated equation is by far the fastest in all cases, see tables 3.1, 3.2 and 3.3. The second order differential equation yields the slowest procedure. This contradicts strongly with the result in [46], where the first procedure (second order differential equation) was the preferred one. In fact, we have strongly simplified the method, allowing shorter calculating times for all three procedures. However simplifications may alter the choice of the most favorable procedure. For the lower order terms the times are comparable, in fact they are practically negligible.

Without pressure terms the calculations are quite fast up to high order, even when the ions are oscillating too (table 3.2). Adding one pressure term increases strongly the times (at least a factor five in the best case): compare table 3.1 with table 3.2. Allowing for both pressure terms once more increases the times strongly: more than an order of magnitude even when taking the simple case when the masses and temperatures of the negative and positive particles are the same (electron-positron case, table 3.3)

The results for cold and electron plasmas (when ions are immobile) are the same as those obtained previously in chapter 2. Apart from reconsidering the cases of cold and electron plasmas, we have also elaborated some nonlinear terms for the case when both the electron and ion pressure are taken into account. The case where both electrons and ions have time to move is relevant to comet tails, fullerenes and electron-positron plasma (pulsar atmospheres). Indeed with temperatures around 10^{10}K (1 MeV) where photons may create electron-positron pairs and where the densities (and thus the frequencies) are extremely high the nonlinear theory may be affected by the magnetic terms.

In sound waves we noticed that the series did not converge at all. The nonlinear approach does not work for sound waves. This is due to the fact that the dispersion relation yields a straight line through the origin ($\omega = v_s\, k$) so that $(\partial^2_{tt} - v_s^2 \Delta)$ is always zero for all (ω, k) and all n: to compensate this the (higher order) coefficient is required to be infinite and thus there can be no convergence. Hence sound waves is one of the cases where our method give bad or no convergence.

Similarly, for the case of ion acoustic waves (when the "*plasma approximation*" was used) we failed to get higher order coefficients. According to Chen, the *plasma approximation* consists of replacing Poisson's equation by $n_+ = n_-$ but maintaining that the divergence of the electric field is different from zero. However, it is clear that this involves an inconsistency, which may sometimes be used for quick, rough results. When going to the nonlinear analysis it is bound to give incorrect results as if effectively turned out in our case. When using the Poisson equation, one is able to determine higher orders.

We have studied these cases where the method applied in this thesis is not suitable rather in detail as it is important to understand the negative results and to know when the method is, not applicable. Yet when dealing with Alfvén waves (again a situation with a dispersion relation corresponding to a straight line passing through origin in the k, ω plane) the outcome will be the opposite of the one here: all higher order waves are allowed, with arbitrary coefficients.

Chapter 4

Perturbation analysis for plasma waves including the magnetic contribution

4.1 Introduction

In the previous chapters we neglected the magnetic contribution. We now expose the theory for plasma including the magnetic effects due to currents created by the motions of the charged particles. Again, our particular interest is on the examination of the coefficients of higher order quantities and hence study the convergence of the series involved.

The present analysis turns out to be more involved as there are now more equations and more variables. The purpose here is to include the magnetic terms and currents involved in the perturbation analysis and compare the results with the previous ones. The main change however, will be that now there are transversal waves (photons) and longitudinal waves (plasma waves) instead of plasma waves only.

4.2 Basis

4.2.1 Model

We consider a plasma which is not subjected to an external magnetic field, in a medium infinite in all directions (no boundary conditions) and at rest. We consider an equilibrium configuration in which (small) perturbations are generated, however, now we will take into consideration the magnetic contribution which was neglected in the previous chapters. No source or sink terms are considered. That means particle creation or recombination or annihilation are not taken into account, or rather that: they compensate each other in equilibrium or in steady state. The remark is particularly important for pulsar atmospheres where the temperatures may rise above 10^{10} K or 1 MeV where photon may disintegrate into an electron - positron pair and inversely, electrons and positrons may recombine to produce photon. Viscosity, resistivity, polarization, external fields or currents, collisions and gravitation are neglected.

4.2.2 Basic Equations

We now use the fluid dynamic equations: (2.1), (2.4) and

$$n_\alpha m_\alpha \left[\partial_t \boldsymbol{v}_\alpha + (\boldsymbol{v}_\alpha \cdot \nabla)\, \boldsymbol{v}_\alpha\right] = q_\alpha n_\alpha \left(\boldsymbol{E} + \boldsymbol{v}_\alpha \times \boldsymbol{B}\right) - \nabla p_\alpha, \tag{4.1}$$

together with the full Maxwell's equations:

$$\nabla \cdot \boldsymbol{B} = 0, \tag{4.2}$$

$$\nabla \cdot \boldsymbol{D} = \sigma = \sum_\alpha n_\alpha q_\alpha, \tag{4.3}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \tag{4.4}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \partial_t \boldsymbol{D}, \tag{4.5}$$

$$\boldsymbol{j} = \sum_\alpha n_\alpha\, q_\alpha\, \boldsymbol{v}_\alpha, \tag{4.6}$$

where $\boldsymbol{B}$ and $\boldsymbol{E}$ are the respective magnetic induction and electric field of the wave, σ and $\boldsymbol{j}$ are the "free" charge and current densities respectively, p_α, n_α, q_α, $\boldsymbol{v}_\alpha$ and m_α are respectively the pressure, the number density, the charge, the velocity and the mass of the particle of the α−th kind of particle species. In addition we assume that the material equations are given by

$$\boldsymbol{B} = \mu\, \boldsymbol{H} \tag{4.7}$$

$$\boldsymbol{D} = \varepsilon\, \boldsymbol{E}, \tag{4.8}$$

where μ is the magnetic permeability and ε is the electric permittivity. Often we may replace μ and ε with their values in vacuum, i.e. μ_0 and ε_0. We take μ and ε as constants, which for homogeneous medium is evident. Of course the wave perturbs the homogeneity and thus μ and ε may undergo small changes. However, we neglect those here.

4.2.3 Equilibrium

Here plasma is considered to be at rest (no oscillation). With a subscript zero again denoting the equilibrium quantities, we put

$$\boldsymbol{v}_{\alpha 0} = \boldsymbol{v}_0 = 0,\ \boldsymbol{j} = 0 \text{ and } \boldsymbol{E}_0 = 0. \tag{4.9}$$

Hence our basic system of equations becomes

$$\nabla \cdot \boldsymbol{H}_0 = 0, \tag{4.10}$$

$$\sum_{\alpha} n_{\alpha\,0} q_\alpha = 0, \tag{4.11}$$

$$\partial_t \boldsymbol{H}_0 = 0, \tag{4.12}$$

$$\nabla \times \boldsymbol{H}_0 = 0, \tag{4.13}$$

$$\nabla p_{\alpha\,0} = 0, \tag{4.14}$$

$$\partial_t n_{\alpha\,0} = 0, \tag{4.15}$$

$$p_{\alpha 0} = K_\alpha \,(n_{\alpha\,0})^{\Gamma_\alpha}. \tag{4.16}$$

We note that $n_{\alpha\,0}$ and $p_{\alpha\,0}$ are constants which are independent of time (t) and space ($\boldsymbol{r}$). As we do not consider here an external magnetic field we put $\boldsymbol{H}_0 = 0$. From (4.11) we get the condition of quasineutrality for plasma in equilibrium.

4.3 Linearization

Here as usual, a small perturbation is introduced in the medium so as to disturb the static equilibrium. This perturbation gives rise to small perturbations of the magnetic field, the mass density and the pressure. Linearizing these quantities (with subscript 1 indicating the first order perturbation quantities) make our basic equations to read

$$\nabla \cdot \boldsymbol{H}_1 = 0, \tag{4.17}$$

$$\varepsilon \nabla \cdot \boldsymbol{E}_1 = \sum_{\alpha} q_\alpha\, n_{\alpha\,1}, \tag{4.18}$$

$$\nabla \times \boldsymbol{E}_1 = -\mu\, \partial_t \boldsymbol{H}_1, \tag{4.19}$$

$$\nabla \times \boldsymbol{H}_1 = \boldsymbol{j}_1 + \varepsilon\, \partial_t \boldsymbol{E}_1, \tag{4.20}$$

$$n_{\alpha\,0}\, m_\alpha\, \partial_t \boldsymbol{v}_{\alpha\,1} = q_\alpha\, n_{\alpha\,0}\, \boldsymbol{E}_1 - \nabla p_{\alpha 1}, \tag{4.21}$$

$$\partial_t n_{\alpha\,1} + \nabla \cdot (n_{\alpha\,0}\, \boldsymbol{v}_{\alpha\,1}) = 0, \tag{4.22}$$

$$p_{\alpha 1} = K_\alpha \Gamma_\alpha n_{\alpha 0}^{\Gamma_\alpha - 1} n_{\alpha 1} = m_\alpha v_{s\alpha}^2 n_{\alpha 1}, \tag{4.23}$$

$$\boldsymbol{j}_1 = \sum_\alpha n_{\alpha 0} \, q_\alpha \, \boldsymbol{v}_{\alpha 1}, \tag{4.24}$$

where $v_{s\alpha}^2$ is given by (3.13). Substituting equation (4.20) into the equation obtained after taking the curl on both sides of (4.19) we obtain

$$\nabla \times (\nabla \times \boldsymbol{E}_1) + c_m^{-2} \partial_{tt}^2 \boldsymbol{E}_1 + \mu \, \partial_t \boldsymbol{j}_1 = 0, \tag{4.25}$$

where $c_m^2 = (\varepsilon \mu)^{-1}$ is the square of the speed of light in the medium. Differentiating equation (4.24) with respect to time we get

$$\partial_t \boldsymbol{j} = \sum_\alpha n_{\alpha 0} \, q_\alpha \, \partial_t \boldsymbol{v}_{\alpha 1}. \tag{4.26}$$

Hence using equations (4.26) and (4.21) in equation (4.25) and applying vector identities, one obtains

$$\nabla (\nabla \cdot \boldsymbol{E}_1) - (\nabla \cdot \nabla) \, \boldsymbol{E}_1 + c_m^{-2} \left(\partial_{tt}^2 \boldsymbol{E}_1 + \omega_p^2 \boldsymbol{E}_1 \right) - \mu \sum_\alpha \frac{q_\alpha}{m_\alpha} \nabla p_{\alpha 1} = 0, \tag{4.27}$$

where

$$\omega_p^2 = \sum_\alpha \omega_\alpha^2 \tag{4.28}$$

is the total plasma frequency and ω_α^2 is given by (3.12). Assuming the perturbed quantities are varying as $e^{i\chi}$, we get from equation (4.27):

$$-\boldsymbol{k} \, (\boldsymbol{k} \cdot \boldsymbol{E}_1) + k^2 \boldsymbol{E}_1 + c_m^{-2} \left(\omega_p^2 - \omega^2 \right) \boldsymbol{E}_1 - i \, \mu \sum_\alpha \frac{q_\alpha}{m_\alpha} \boldsymbol{k} \, p_{\alpha 1} = 0. \tag{4.29}$$

Notice that this equation does not split into a real and an imaginary part. This is so because $\boldsymbol{E}_1$, as is clear from equation (4.18), contains the imaginary unit i with respect to $n_{\alpha 1}$, $p_{\alpha 1}$ and $\boldsymbol{v}_{\alpha 1}$ i.e. the phase of $\boldsymbol{E}_1$ is $\pi/2$ different from $n_{\alpha 1}$, $p_{\alpha 1}$ and $\boldsymbol{v}_{\alpha 1}$.

Taking ∂_t on both sides of equation (4.23) and substituting equation (4.22) into the obtained result, we have

$$\partial_t p_{\alpha 1} = -m_\alpha v_{s\alpha}^2 \, \mathrm{div} \, (n_{\alpha 0} \, \boldsymbol{v}_{\alpha 1}) \, . \tag{4.30}$$

We then differentiate (4.30) with respect to t and substitute (4.21) into into get:

$$\partial_{tt}^2 p_{\alpha 1} = -v_{s\alpha}^2 \, \mathrm{div} \, (q_\alpha n_{\alpha 0} \, \boldsymbol{E}_1 - \nabla p_{\alpha 1}) \, . \tag{4.31}$$

Using again $\exp(i \, \chi)$ as dependence for the first order quantities yields

$$p_{\alpha 1} = \frac{i \, q_\alpha \, v_{s\alpha}^2 n_{\alpha 0} \, (\boldsymbol{k} \cdot \boldsymbol{E}_1)}{\omega^2 - k^2 v_{s\alpha}^2}. \tag{4.32}$$

Plugging this equation into equation (4.29) gives

$$k^2\boldsymbol{E}_1+c_m^{-2}\left(\omega_p^2-\omega^2\right)\boldsymbol{E}_1-\left(1-c_m^{-2}\sum_\alpha\frac{\omega_\alpha^2\,v_{s\alpha}^2}{\omega^2-k^2\,v_{s\alpha}^2}\right)\boldsymbol{k}\left(\boldsymbol{k}\cdot\boldsymbol{E}_1\right)=0. \quad (4.33)$$

Let us decompose $\boldsymbol{E}_1$ in a component parallel to $\boldsymbol{k}$ ($\boldsymbol{E}_{1\|}$) and one perpendicular to $\boldsymbol{k}$ ($\boldsymbol{E}_{1\perp}$). We obtain from (4.33)

$$\left[k^2+c_m^{-2}\left(\omega_p^2-\omega^2\right)\right]\boldsymbol{E}_{1\|}-\left(1-c_m^{-2}\sum_\alpha\frac{\omega_\alpha^2\,v_{s\alpha}^2}{\omega^2-k^2v_{s\alpha}^2}\right)\boldsymbol{k}\left(\boldsymbol{k}\cdot\boldsymbol{E}_{1\|}\right)=0. \quad (4.34)$$

and

$$k^2\boldsymbol{E}_{1\perp}+c_m^{-2}\left(\omega_p^2-\omega^2\right)\boldsymbol{E}_{1\perp}=0. \quad (4.35)$$

Hence, we have essentially only two cases: transversal waves ($\boldsymbol{k}\cdot\boldsymbol{E}_1=0$) i.e photons, and longitudinal waves ($\boldsymbol{k}\parallel\boldsymbol{E}$) i.e. plasma waves.

4.3.1 Transverse waves ($\boldsymbol{k}\perp\boldsymbol{E}_1 \Longrightarrow \boldsymbol{k}\cdot\boldsymbol{E}_{1\perp}=0$)

4.3.1.1 Dispersion relation

If $\boldsymbol{E}_1\neq 0$, then equation (4.35) yields the well-known dispersion relation for electromagnetic waves propagating in a plasma with no external magnetic field:

$$\omega^2=\omega_p^2+k^2\,c_m^2. \quad (4.36)$$

This shows that the presence of the pressure terms (i.e. thermal motion of particles) in the basic equations do not affect the dispersion relation for electromagnetic waves. This is due to the fact that p_α has to be zero, in the view of equation (4.32) (with $\omega^2\neq k^2v_{s\alpha}^2$ of course).

Comparing (4.36) with the dispersion relation for the light waves in vacuum:

$$\omega^2=k^2\,c^2, \quad (4.37)$$

and neglecting the possible difference between c and c_m, one notices that equation (4.37) is being modified by a term ω_p^2. This term therefore acts as a cut-off frequency (for most of laboratory plasmas, the cut-off frequency lies in the microwave range [32, p. 116]), as it prevents the waves with frequencies below it to propagate. In fact for low frequencies (below cut-off) the electrons move fast enough as a whole so that they are able to counteract the wave field completely, thus annihilating it in a very short time and thus keeping the quasi-neutrality and the current free situation.

4.3.1.2 Determination of the expressions for the linearized quantities

It is evident from equation (4.32) that

$$p_{\alpha 1} = 0, \tag{4.38}$$

and as a consequence of this

$$n_{\alpha 1} = 0. \tag{4.39}$$

From equation (4.21) one obtains:

$$\boldsymbol{E}_1 = \boldsymbol{E}_{1\perp} = i\,\frac{\omega\, m_\alpha}{q_\alpha}\,\boldsymbol{v}_{\alpha 1}. \tag{4.40}$$

Putting (4.40) into (4.24) we have:

$$\boldsymbol{j}_1 = -\frac{i\,\boldsymbol{E}_{1\perp}}{\omega}\sum_\alpha \frac{n_{\alpha 0}\, q_\alpha^2}{m_\alpha}. \tag{4.41}$$

For the case of magnetic induction we get from equation (4.19):

$$\boldsymbol{H}_1 = \boldsymbol{H}_{1\perp} = -\frac{1}{\mu\,\omega}(\boldsymbol{k} \times \boldsymbol{E}_{1\perp}) \tag{4.42}$$

Hence, by fixing the amplitude of one of the quantities, say $\boldsymbol{E}_1$, one is able to determine all other quantities ($\boldsymbol{v}_{\alpha 1}$, $\boldsymbol{j}_1$, $\boldsymbol{H}_1$). Note that $\boldsymbol{E}_1$, $\boldsymbol{v}_{\alpha 1}$, $\boldsymbol{j}_1$ and $\boldsymbol{H}_1$ all lay in the plane perpendicular to the propagation of the wave: then the wave fully deserves the name: transversal.

4.3.2 Longitudinal waves ($\boldsymbol{k} \parallel \boldsymbol{E}_1 \Longrightarrow \boldsymbol{k}\cdot\boldsymbol{E}_{1\parallel} = kE_{1\parallel}$)

4.3.2.1 Dispersion relation

Here we take a dot product between $\boldsymbol{k}$ and equation (4.34) to have:

$$\omega^2 = \omega_p^2 + \sum_\alpha \frac{\omega_\alpha^2\, k^2\, v_{s\alpha}^2}{\omega^2 - k^2 v_{s\alpha}^2}. \tag{4.43}$$

Depending on the choice of α, we may determine from equation (4.43) various dispersion relations seen before. For example, if one considers the two component cold plasma case, one obtains again from (4.43) the well-known dispersion relation for the cold plasma with no external magnetic field (cf. equation (3.25)). And from the same equation (4.43), when one considers the case of electron plasma, one recovers the Langmuir dispersion relation together with the trivial solution ($\omega^2 = 0$). For the case of electron-positron plasma one recovers the dispersion relations given in equation (3.30). In fact, equation (4.43) is just the generalization of equation (3.27) now with α instead of $\pm$ only. By substituting equation (4.28) into (4.43) one obtains $\omega^2 = 0$ and

$$\sum_\alpha \frac{\omega_\alpha^2}{\omega^2 - k^2 v_{s\alpha}^2} = 1. \tag{4.44}$$

The latter is in the form of the dispersion relation which can easily be compared with equation (3.27).

4.3.2.2 Determination of the first order perturbation quantities

Consider equation (4.21) and take the divergence on both of its sides. Into the determined result, substitute equations (4.18), (4.22) and (4.23) in order to eliminate $\boldsymbol{E}_{1\|}$, $\boldsymbol{v}_{\alpha 1}$ and $p_{\alpha 1}$. Assuming once more that all quantities are functions which are varying as $e^{i\chi}$, we get (with $\beta = \alpha$):

$$\omega^2 n_{\alpha 1} = k^2\, v_{s\alpha}^2\, n_{\alpha 1} + \frac{\omega_\alpha^2}{q_\alpha} \sum_\beta q_\beta\, n_{\beta 1}, \tag{4.45}$$

which are α linear and homogeneous equations. From their determinant one may recover the dispersion relation. Next one can determine the first order expressions for the particle densities using one arbitrary coefficient. For the illustration purposes, we consider the cold plasma case (both ions and electron oscillating). Substituting $n_{\alpha 1} = n_{\pm 1} = c_{\pm 1\rho}\, n_{\pm 0}\, A \exp(i\chi)$ (with $c_{-1\rho} = 1$ and $\omega^2 = \omega_+^2 + \omega_-^2$) into equation (4.45) we obtain $c_{+1\rho} = -\omega_+^2/\omega_-^2$ which is exactly the same as the one given in appendix B.2. For completeness, let us give explicitly the expressions for the first order physical quantities:

$$n_{-1} = n_0\, A\, e^{i\chi} \quad \text{and} \quad n_{+1} = -\frac{\omega_+^2\, n_{-1}}{\omega_-^2}, \tag{4.46}$$

$$\boldsymbol{v}_{-1} = -\frac{\omega n_{-1}}{n_0\, k^2}\boldsymbol{k} \quad \text{and} \quad \boldsymbol{v}_{+1} = \frac{\omega\, \omega_+^2\, n_{-1}}{k^2\, \omega_-^2\, n_0}\boldsymbol{k}, \tag{4.47}$$

$$\boldsymbol{j}_1 = \left(1 + \frac{\omega_+^2}{\omega_-^2}\right) \frac{e\, \omega\, n_{-1}\, \boldsymbol{k}}{k^2}, \tag{4.48}$$

$$\boldsymbol{E}_1 = \boldsymbol{E}_{1\|} = i\, \frac{\omega\, m_\pm}{q_\pm}\, v_{\pm 1} = \frac{i\, m_\pm\, \omega^2 \omega_\pm^2\, n_{-1}}{e\, k^2\, \omega_-^2\, n_0}\boldsymbol{k} = \frac{i\, m_-\, \omega^2\, n_{-1}}{e\, k^2\, n_0}\boldsymbol{k}. \tag{4.49}$$

$$\boldsymbol{H}_1 = \boldsymbol{H}_{1\|} = -\frac{1}{\mu\,\omega}(\boldsymbol{k} \times \boldsymbol{E}_1) = 0. \tag{4.50}$$

These are the quantities which are involved in our linearized two component cold plasma. They will later on be used in the determination of the coefficients of higher order terms; in one or two component cold plasma case.

4.4 Higher orders

Assuming again that all quantities are function of χ alone, reduces our basic system of equations (i.e. equations (2.1), (2.4) and (4.1) - (4.8)) to the following general system equations:

$$\boldsymbol{k}\cdot\boldsymbol{H}'=0 \quad \text{or} \quad \boldsymbol{k}\cdot\boldsymbol{H}=0, \tag{4.51}$$

$$\varepsilon\boldsymbol{k}\cdot\boldsymbol{E}'=\sum_\alpha q_\alpha n_\alpha \quad \text{or} \quad \varepsilon\boldsymbol{k}\cdot\boldsymbol{E}=\sum_\alpha\int q_\alpha n_\alpha, \tag{4.52}$$

$$\boldsymbol{k}\times\boldsymbol{E}'=-\mu\omega\boldsymbol{H}' \quad \text{or} \quad \boldsymbol{k}\times\boldsymbol{E}=-\mu\omega\boldsymbol{H}, \tag{4.53}$$

$$\boldsymbol{k}\times\boldsymbol{H}'=\boldsymbol{j}+\varepsilon\omega\boldsymbol{E}' \quad \text{or} \quad \boldsymbol{k}\times\boldsymbol{H}=\int\boldsymbol{j}+\varepsilon\omega\boldsymbol{E}, \tag{4.54}$$

$$m_\alpha n_\alpha\left[\omega+(\boldsymbol{v}_\alpha\cdot\boldsymbol{k})\right]\boldsymbol{v}_\alpha'=q_\alpha n_\alpha\left(\boldsymbol{E}+\boldsymbol{v}_\alpha\times\mu\boldsymbol{H}\right)-\boldsymbol{k}p_\alpha', \tag{4.55}$$

$$\omega n_\alpha'+\boldsymbol{k}\cdot(n_\alpha\boldsymbol{v}_\alpha)'=0, \tag{4.56}$$

$$p_\alpha=K_\alpha n_\alpha^{\Gamma_\alpha}. \tag{4.57}$$

$$\boldsymbol{j}=\sum_\alpha q_\alpha n_\alpha\boldsymbol{v}_\alpha. \tag{4.58}$$

As we did previously, we again decompose $\boldsymbol{E}$ into components parallel and perpendicular to $\boldsymbol{k}$. Doing so leads us into considering the the longitudinal and transversal cases:

4.4.1 Longitudinal case

4.4.1.1 Second order

From the above system of equations we have

$$\varepsilon\boldsymbol{k}\cdot\boldsymbol{E}_2'=\sum_\alpha q_\alpha n_{\alpha 2} \quad \text{or} \quad \varepsilon\boldsymbol{k}\cdot\boldsymbol{E}_2=\sum_\alpha\int q_\alpha n_{\alpha 2}\,d\chi, \tag{4.59}$$

$$\boldsymbol{H}_2=0, \tag{4.60}$$

$$0=\boldsymbol{j}_2+\varepsilon\omega\boldsymbol{E}_2' \quad \text{or} \quad 0=\int\boldsymbol{j}_2\,\mathrm{d}\chi+\varepsilon\omega\boldsymbol{E}_2, \tag{4.61}$$

$$n_{\alpha 0}\,m_\alpha\left[\omega\boldsymbol{v}_{\alpha 2}'+(\boldsymbol{v}_{\alpha 1}\cdot\boldsymbol{k})\,\boldsymbol{v}_{\alpha 1}'\right]+n_{\alpha 1}\,m_\alpha\,\omega\boldsymbol{v}_{\alpha 1}'$$

$$= q_\alpha n_{\alpha 0} \left(\boldsymbol{E}_2 + \boldsymbol{v}_{\alpha 1} \times \mu \boldsymbol{H}_1 \right) + q_\alpha n_{\alpha 1} \boldsymbol{E}_1 - \boldsymbol{k} p'_{\alpha 2}, \tag{4.62}$$

$$\omega n'_{\alpha 2} + \boldsymbol{k} \cdot (n_{\alpha 0} \boldsymbol{v}_{\alpha 2})' + \boldsymbol{k} \cdot (n_{\alpha 1} \boldsymbol{v}_{\alpha 1})' = 0 \quad \text{or}$$

$$\omega n'_{\alpha 2} + n_{\alpha 0} \boldsymbol{k} \cdot \boldsymbol{v}'_{\alpha 2} + n_{\alpha 1} \boldsymbol{k} \cdot \boldsymbol{v}'_{\alpha 1} + n'_{\alpha 1} \boldsymbol{k} \cdot \boldsymbol{v}_{\alpha 1} = 0, \tag{4.63}$$

$$p_{\alpha 2} = m_\alpha v_{s\alpha}^2 \left(n_{\alpha 2} + \frac{(\Gamma_\alpha - 1)\, n_{\alpha 1}^2}{2 n_{\alpha 0}} \right), \tag{4.64}$$

$$\boldsymbol{j}_2 = \sum_\alpha q_\alpha \left(n_{\alpha 0} \boldsymbol{v}_{\alpha 2} + n_{\alpha 1} \boldsymbol{v}_{\alpha 1} \right). \tag{4.65}$$

Determination of $n_{\alpha 2}$

Differentiate equation (4.62) with respect to χ and then take a dot product with $\boldsymbol{k}$ to have

$$n_{\alpha 0} m_\alpha \boldsymbol{k} \cdot \left\{ \omega \boldsymbol{v}''_{\alpha 2} + \left[(\boldsymbol{v}_{\alpha 1} \cdot \boldsymbol{k}) \boldsymbol{v}'_{\alpha 1} \right]' \right\} + n_{\alpha 1} m_\alpha \omega \boldsymbol{k} \cdot \boldsymbol{v}''_{\alpha 1} + n'_{\alpha 1} m_\alpha \omega \boldsymbol{k} \cdot \boldsymbol{v}'_{\alpha 1}$$

$$= q_\alpha n_{\alpha 0} \boldsymbol{k} \cdot \left[\boldsymbol{E}'_2 + (\boldsymbol{v}_{\alpha 1} \times \mu \boldsymbol{H}_1)' \right] + q_\alpha n_{\alpha 1} \boldsymbol{k} \cdot \boldsymbol{E}'_1 + q_\alpha n'_{\alpha 1} \boldsymbol{k} \cdot \boldsymbol{E}_1 - k^2 p''_{\alpha 2}. \tag{4.66}$$

We then eliminate $\boldsymbol{E}_2$, $\boldsymbol{v}_{\alpha 2}$ and $p_{\alpha 2}$ by substituting equations (4.59), (4.63) and (4.64) into equation (4.66):

$$\left(k^2 v_{s\alpha}^2 - \omega^2 \right) n''_{\alpha 2} - \frac{\omega_\alpha^2}{q_\alpha} \sum_\beta q_\beta n_{\beta 2}$$

$$+ n_{\alpha 0} \boldsymbol{k} \cdot \left[(\boldsymbol{v}_{\alpha 1} \cdot \boldsymbol{k}) \boldsymbol{v}'_{\alpha 1} \right]' + n_{\alpha 1} \omega \boldsymbol{k} \cdot \boldsymbol{v}''_{\alpha 1} + n'_{\alpha 1} \omega \boldsymbol{k} \cdot \boldsymbol{v}'_{\alpha 1}$$

$$= \omega \left(n_{\alpha 1} \boldsymbol{k} \cdot \boldsymbol{v}''_{\alpha 1} + 2 n'_{\alpha 1} \boldsymbol{k} \cdot \boldsymbol{v}'_{\alpha 1} + n''_{\alpha 1} \boldsymbol{k} \cdot \boldsymbol{v}_{\alpha 1} \right) + \frac{q_\alpha n_{\alpha 1}}{m_\alpha} \boldsymbol{k} \cdot \boldsymbol{E}'_1$$

$$+ \frac{q_\alpha n'_{\alpha 1} \boldsymbol{k} \cdot \boldsymbol{E}_1}{m_\alpha} - \frac{(\Gamma_\alpha - 1)\, k^2 v_{s\alpha}^2 \left(n_{\alpha 1} n''_{\alpha 1} + n'^2_{\alpha 1} \right)}{n_{\alpha 0}} \tag{4.67}$$

with $\beta = \alpha$. Assuming perturbed quantities depend on $e^{i\chi}$ (for the first order quantities) and $e^{2i\chi}$ (for the second order quantities), we get from equation (4.67), the following:

$$2 \left(\omega^2 - k^2 v_{s\alpha}^2 \right) n_{\alpha 2} - \frac{\omega_\alpha^2}{2 q_\alpha} \sum_\beta q_\beta n_{\beta 2} = n_{\alpha 0} \boldsymbol{k} \cdot (\boldsymbol{v}_{\alpha 1} \cdot \boldsymbol{k}) \boldsymbol{v}_{\alpha 1}$$

$$- n_{\alpha 1} \omega \boldsymbol{k} \cdot \boldsymbol{v}_{\alpha 1} + i \frac{q_\alpha n_{\alpha 1}}{m_\alpha} \boldsymbol{k} \cdot \boldsymbol{E}_1 + \frac{(\Gamma_\alpha - 1)\, k^2 v_{s\alpha}^2 n_{\alpha 1}^2}{n_{\alpha 0}}. \tag{4.68}$$

The left hand side corresponds to equation (4.45), with ω, $n_{\alpha 1}$ and k now replaced by 2ω, $n_{\alpha 2}$ and $2k$ as it should. The right hand side is constituted by terms quadratic in the first order quantities, and thus known. Hence, from the linear set of equations one may determine all $n_{\alpha 2}$. For example for the case of electron-ion plasma, one gets two equations from equation (4.68). Solving these equations simultaneously, one obtains the coefficient $c\pm 2$ given in appendix B.3.

Determination of $v_{\alpha 2}$

We recall that

$$n_\alpha (\omega + \boldsymbol{k} \cdot \boldsymbol{v}_\alpha) = \omega n_{\alpha 0} = \in . \tag{4.69}$$

Hence for second order, the following expression for the perturbed velocity is obtained:

$$\boldsymbol{v}_{\alpha 2} = -\left(\frac{n_{\alpha 1}}{n_{\alpha 0}} \boldsymbol{v}_{\alpha 1} + \frac{\omega n_{\alpha 2}}{n_{\alpha 0} k^2} \boldsymbol{k}\right) \tag{4.70}$$

assuming that $\boldsymbol{v}_{\alpha 2}$ also is parallel to $\boldsymbol{k}$ just as $\boldsymbol{v}_{\alpha 1}$. Substituting the appropriate quantities in equation (4.70) yields the coefficients of the second order expressions for the velocity (see appendix B.3). It is also evident that equation (4.70) satisfies the continuity equation (4.63).

Determination of $\boldsymbol{E}_{2\|}$ and $\boldsymbol{j}_2$

Using equations (4.64) and (4.70) in equation (4.62), eliminates $p_{\alpha 2}$ and $\boldsymbol{v}_{\alpha 2}$. Then assuming $e^{i\chi}$ (for first order) and $e^{2i\chi}$ (for second order) dependencies, we have

$$\boldsymbol{E}_{2\|} = -\frac{n_{\alpha 1}}{n_{\alpha 0}} \boldsymbol{E}_{1\|} + i \left\{ 2k \left(v_{s\alpha}^2 - \frac{m_\alpha \omega^2}{q_\alpha n_{\alpha 0} k^2} \right) n_{\alpha 2} \right.$$

$$\left. - \frac{m_\alpha}{q_\alpha n_{\alpha 0}} \left[\Big(\omega n_{\alpha 1} - n_{\alpha 0} (\boldsymbol{v}_{\alpha 1} \cdot \boldsymbol{k}) \Big) \boldsymbol{v}_{\alpha 1} - \frac{(\Gamma_\alpha - 1)\, \boldsymbol{k}\, v_{s\alpha}^2 n_{\alpha 1}^2}{n_{\alpha 0}} \right] \right\}. \tag{4.71}$$

Plugging the suitable quantities into equation (4.71), one gets the expression of $\boldsymbol{E}_{2\|}$. For the case of current we put equation (4.70) into equation (4.65) to have

$$\boldsymbol{j}_2 = -\frac{\omega}{k^2} \boldsymbol{k} \sum_\alpha q_\alpha n_{\alpha 2}. \tag{4.72}$$

Substituting for $n_{\alpha 2}$ one gets the explicit expression of $\boldsymbol{j}_2$. On the other hand we notice that equation (4.72) agrees exactly with equation (4.61). This is quite satisfactory as we have two expressions for $\boldsymbol{j}_2$ and they have to be consistent. In fact, due to the equation of charge conservation this is practically guaranteed.

Example: The expressions for the second order quantities in cold plasma case (ions stand still)

Using equations (4.46) - (4.49) into equations (4.68) - (4.71) we obtain the expressions of n_{-2}, $\boldsymbol{v}_{-2}$, $E_{2\|}$ and hence that of $\boldsymbol{j}_2$. These may explicitly be written as:

$$n_{-2} = 2\frac{n_{-1}^2}{n_0} \tag{4.73}$$

$$\boldsymbol{v}_{-2} = -\frac{\omega\, n_{-1}^2}{n_0^2\, k^2}\boldsymbol{k} \tag{4.74}$$

$$\boldsymbol{E}_{2\|} = \frac{i\, m_-\, \omega^2\, n_{-1}^2}{e\, n_0^2\, k^2}\boldsymbol{k} \tag{4.75}$$

$$\boldsymbol{j}_2 = \frac{\omega\, e\, n_{-2}}{k^2}\boldsymbol{k} = \frac{2\,\omega\, e\, n_{-1}^2}{k^2\, n_0}\boldsymbol{k} = -\varepsilon\,\omega \boldsymbol{E}_2' \tag{4.76}$$

$$\boldsymbol{H}_2 = -\frac{1}{\mu\,\omega}\,(\boldsymbol{k}\times\boldsymbol{E}_2) = 0. \tag{4.77}$$

As a matter of completion, we also consider the arbitrary order case.

4.4.1.2 n^{th} order

Equation (4.54) yields

$$\boldsymbol{j} = -\varepsilon\omega\boldsymbol{E}_{\|}' \tag{4.78}$$

since from equation (4.53) we have $\boldsymbol{H}_{\|} = 0$. This indicates clearly that the magnetic field therefore plays no role to the longitudinal waves. Thus the analysis here has to correspond to the one considered previously. To verify this, let us put (4.69) into (4.55) and then take a dot product with $\boldsymbol{k}$ followed by the elimination of $\boldsymbol{v}_\alpha$ and p_α using equations (4.56) and (4.57). This yields

$$-\frac{m_\alpha\,\omega^2\, n_{\alpha 0}^2\, n_\alpha'}{n_\alpha^3} = q_\alpha\, \boldsymbol{k}\cdot\boldsymbol{E}_{\|} - k^2\, K_\alpha\, \Gamma_\alpha\, n_\alpha^{\Gamma_\alpha - 2} n_\alpha'. \tag{4.79}$$

Elimination of $\boldsymbol{k}\cdot\boldsymbol{E}_{\|}$ by using equation (4.52) leads one to the recovery of the nonlinear differential equation of second order (cf. equations (3.11)) or of the nonlinear differential equation of first order (cf. equation (3.14)). Integrating equation (3.14) we recover the fully integrated equation (cf. equation (3.15)). Hence the results for the longitudinal case, are the same as those determined in the previous chapters. But in addition, we are now able to write explicitly the higher order expressions for $\boldsymbol{j}$ and $\boldsymbol{E}_{\|}$. And for the magnetic field we have $\boldsymbol{H} = 0$.

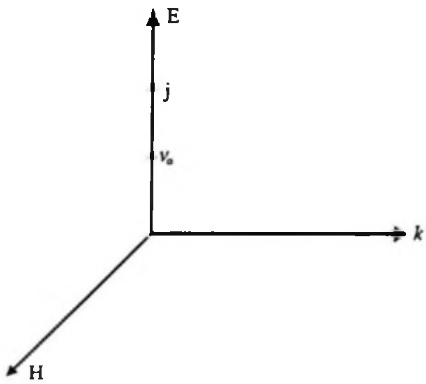

Figure 4.1: *Transversal case: the figure shows that $\boldsymbol{k}$ points in the direction which is perpendicular to both $\boldsymbol{H}$ and $\boldsymbol{E}$ and that $\boldsymbol{E}$ is parallel to both $\boldsymbol{j}$ and $\boldsymbol{v}_\alpha$.*

4.4.2 Transversal case

It is difficult to determine the higher order expressions for the case of transversal waves. But if one assumes that $\boldsymbol{k}\cdot\boldsymbol{v}_\alpha = 0$ (see figure 4.1) then one may be able to fix the higher order expressions as we shall see below:

4.4.2.1 Second order (with $\boldsymbol{k}\cdot\boldsymbol{v}_\alpha = 0$)

From equations (4.51) - (4.54) we have

$$\boldsymbol{k}\cdot\boldsymbol{H}'_{2\perp} = 0 \quad \text{or} \quad \boldsymbol{k}\cdot\boldsymbol{H}_{2\perp} = 0, \tag{4.80}$$

$$\varepsilon\boldsymbol{k}\cdot\boldsymbol{E}'_{2\perp} = 0 = \sum_\alpha q_\alpha n_{\alpha 2} \quad \text{or} \quad \varepsilon\boldsymbol{k}\cdot\boldsymbol{E}_{2\perp} = 0 = \sum_\alpha \int q_\alpha n_{\alpha 2}\, d\chi, \tag{4.81}$$

$$\boldsymbol{k}\times\boldsymbol{E}'_{2\perp} = -\mu\omega\boldsymbol{H}'_{2\perp} \quad \text{or} \quad \boldsymbol{k}\times\boldsymbol{E}_{2\perp} = -\mu\omega\boldsymbol{H}_{2\perp}, \tag{4.82}$$

$$\boldsymbol{k}\times\boldsymbol{H}'_{2\perp} = \boldsymbol{j}_2 + \varepsilon\omega\boldsymbol{E}'_{2\perp} \quad \text{or} \quad \boldsymbol{k}\times\boldsymbol{H}_{2\perp} = \int \boldsymbol{j}_2\, d\chi + \varepsilon\omega\boldsymbol{E}_{2\perp}. \tag{4.83}$$

From (4.56) we obtain:

$$\epsilon_\alpha = \omega n_{\alpha 0} = (n_{\alpha 0} + n_{\alpha 1} + n_{\alpha 2})\left[\omega + \boldsymbol{k}\cdot(\boldsymbol{v}_{\alpha 1} + \boldsymbol{v}_{\alpha 2})\right]. \tag{4.84}$$

But since $n_{\alpha 1} = 0$ (see equation (4.39)) and $\boldsymbol{k}\cdot\boldsymbol{v}_{\alpha 2} = 0$, equation (4.84) becomes

$$n_{\alpha 2} = 0 \tag{4.85}$$

and hence, equation (4.57) gives:

$$p_{\alpha 2} = v_{s\alpha}^2 n_{\alpha 2}^{\Gamma_\alpha} = 0 \tag{4.86}$$

Using these into equations (4.55) and (4.58), we respectively obtain:

$$m_\alpha \omega \boldsymbol{v}'_{\alpha 2} = q_\alpha \left(\boldsymbol{E}_{2\perp} + \boldsymbol{v}_{\alpha 1} \times \mu \boldsymbol{H}_{1\perp} \right) \tag{4.87}$$

and

$$\boldsymbol{j}_2 = \sum_\alpha q_\alpha n_{\alpha 0} \boldsymbol{v}_{\alpha 2}. \tag{4.88}$$

Then by using equations (4.82) and (4.88), we eliminate $\boldsymbol{H}_2$ and $\boldsymbol{j}_2$ from equation (4.83) to have

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}'_{2\perp}) = -\mu\omega \left(\sum_\alpha q_\alpha n_{\alpha 0} \boldsymbol{v}_{\alpha 2} + \varepsilon \omega \boldsymbol{E}'_{2\perp} \right). \tag{4.89}$$

Applying vector identities to (4.89) and eliminating $\boldsymbol{v}_{\alpha 2}$ using equation (4.87), followed by plugging equations (4.40) and (4.42) into the resulted equation, yields

$$k^2 \boldsymbol{E}''_{2\perp} - \mu\varepsilon \left\{ \omega^2 \boldsymbol{E}''_{2\perp} + \omega_p^2 \boldsymbol{E}_{2\perp} + \frac{i}{\omega^2} \sum_\alpha \frac{\omega_\alpha^2 q_\alpha^2}{m_\alpha} \left[\boldsymbol{E}_{1\perp} \times (\boldsymbol{k} \times \boldsymbol{E}_{1\perp}) \right] \right\} = 0. \tag{4.90}$$

Since $\boldsymbol{E}_{1\perp}$ was fixed in the first order case, then one may use equation (4.90) in the determination of $\boldsymbol{E}_{2\perp}$ explicitly. Other quantities are then determined.

4.5 Conclusion

We have investigated the case of plasma including the magnetic contribution due to the perturbation. As compared to the previous case (plasma without the magnetic terms) where only longitudinal waves occurred, now we have two clear distinct cases: transversal waves and longitudinal ones.

For the case of longitudinal waves, our system of equations was reduced to the equations obtained in the preceding chapters. The addition equations turn out to be consistent with the previous ones. Hence the results are the same as those obtained earlier and in addition we have now written explicitly the expressions for higher orders of the current density. The magnetic induction remains zero even for the perturbed quantities. The convergence analysis here is obviously unchanged.

For the transversal waves, we were able to recover the dispersion relation for electromagnetic waves propagating in plasma with no external magnetic field. The particle density and pressure remain now constant in the perturbation. For the first order of the velocity, electric field and current density, one has to fix one of the quantities, say the amplitude of the electric field, in order to determine the other quantities.

Chapter 5

Stability of uniform gravitational medium with cosmological constant

5.1 Introduction

Jeans [3, 4] applied the linearized perturbation theory developed by Lord Rayleigh [2] to an infinite homogeneous gravitating medium at rest. This perturbation analysis was all right but Jeans started from an equilibrium which was impossible in the Newtonian gravitation. The difficulty is not a trivial one and continued to raise criticism of the so called Jeans's instability criterion see e.g. Čadež and Verheest [50]. In fact Einstein faced in 1917 the same difficulty concerning the equilibrium when he tried to apply his monumental theory of gravitation (usually called the general theory of relativity) to cosmology. Even the introduction of curvature did not eliminate the basic obstacle: the only possible homogeneous and static universe allowed by his original equations of 1916 had to be an empty one (devoid of mass or energy: i.e. density $\rho_0 = 0$). To avoid the empty universe, Einstein introduced in 1917 the so called cosmological term with the cosmological constant Λ in his theory of gravitation [51, 52]. This was the only possible generalization within his scheme [53]. Various studies show that Λ is extremely small [53, 54] and its influence in the solar system for example can not yet be measured [40]. Einstein himself objected very much the cosmological term as it looks like introducing a negative mass density for which he did not see a physical counterpart in the real world. (In fact for a plasma the problem does not occur as positive and negative charges exist and a quasi-static equilibrium in which both charges compensate each other is a trivial reality.) Einstein later on removed this term after realizing that the universe is expanding in view of the observation of receding galaxies by Hubble [55] and the theory by Lemaître [56]. On the expanding cosmos starting from the "primitive atom" (huge as an "atom", but small as a universe) as a forerunner of the big bang theory. Recently observations of supernovae type 1 (used as standard candles, allowing to look back far in the history of the cosmos) indicate that Λ might even have the opposite sign of the one expected and that even an accelerated expansion is possible, (see for example [53]). By all means the situation is far from clear and the interest in the cosmological term is revived strongly.

In this study we are looking for a nonlinear stability analysis in the Newtonian approximation, however with a cosmological constant.

In fact there is a fair similarity between the plasma case and the gravitational one and the differences and similarities between both analyses turn out quite interesting. Moreover, before getting involved in relativistic corrections and the much more complicated calculations one should have a clear cut idea of what may be expected on the basis of a Newtonianlike approximation.

5.2 Basis

5.2.1 Model

We consider a gravitating medium at rest which is homogeneous and infinite in all directions. We work with the Newtonian approximation in which is introduced a cosmological constant, thus allowing a sound equilibrium. Viscosity and electromagnetic contributions are neglected.

5.2.2 Basic equations

The basic equations are the continuity equation, the equation of motion, the Poisson equation (i.e. the field equation for Newtonian gravitation) with a cosmological constant and the polytropic equation. Thus, the basic system of equations is written [12]

$$\partial_t \rho + div(\rho v) = 0, \tag{5.1}$$

$$\rho \frac{dv}{dt} = -\nabla p - \rho \nabla \varphi, \tag{5.2}$$

$$\Delta \varphi + \Lambda \varphi = 4\pi G \rho, \tag{5.3}$$

$$p = K \rho^{\Gamma}, \tag{5.4}$$

where $\rho(\boldsymbol{r},t)$ is the mass density, $\boldsymbol{v}(\boldsymbol{r},t)$ is the velocity, $\varphi(\boldsymbol{r},t)$ is the gravitational potential, Λ is the cosmological constant, G is the gravitational constant ($\approx$ $(6.6726 \pm 0.0005) \times 10^{-11}\mathrm{m}^3/\mathrm{kg\ s}^2$), $p(\boldsymbol{r},t)$ is the pressure, K and Γ are constants, $\boldsymbol{r}$ is the space and t is the time.

Equation (5.3) is an approximation like the one leading to the Newtonian theory from Einstein's gravitational field equation of 1917, where he introduced the cosmological term. In fact the potential corresponding to equation (5.3) for a point mass m situated at $r = 0$ of a spherical coordinate system is:

$$\varphi = -\frac{G\,m\,e^{-\sqrt{-\Lambda}r}}{r}. \tag{5.5}$$

With $\Lambda = 0$ one recovers the well-known Newtonian potential (up to an arbitrary constant). The exponential was first introduced by Neumann [57] in the 19th century (cf. the Debye and Yukawa potential). It weakens the r^{-1} potential. At great distances ($|\Lambda|$ very small, i.e. cosmological distances, $r >> \sqrt{|\Lambda|}$) it prevents the potential of a (uniform) medium where the volume approaches infinity, from becoming infinite.

E.g. a uniform sphere with mass $4\pi\rho r^3/3$ has a potential $\varphi = -4\pi G\rho r^2/3$ at its boundary and $2\pi G\rho r^2$ in its center, up to an arbitrary constant.

One may interpret equation (5.5) as the result of two factors:

- The spherical expansion (like light) yielding the factor r^{-2} in the field $(-\nabla\varphi)$ and thus r^{-1} in the potential as in Newtonian gravitation.
- An "absorption", obviously leading to an exponential. However, it is not clear why the absorption should be proportional to r: we rather expect it proportional to the amount of mass crossed up to the distance r. However, in a universe of uniform density this may work nevertheless all right: Λ may then be proportional to the uniform density, but it is constant all the same (at least in the equilibrium)

5.3 Analysis of perturbations

5.3.1 Equilibria with a constant gravitational potential or homogeneous media

With $\boldsymbol{v}_0 = 0$ and either homogeneous pressure or φ_{g0} = constant, the basic equations give for the equilibrium quantities

$$\partial_t\rho_{g0} = 0, \tag{5.6}$$

$$\nabla p_{g0} = -\rho_{g0} \nabla \varphi_{g0} = 0, \tag{5.7}$$

$$\Lambda \varphi_{g0} = 4\pi G \rho_{g0}, \tag{5.8}$$

$$p_{g0} = K \rho_{g0}^{\Gamma}. \tag{5.9}$$

Using equations (5.6) - (5.9) we see that ρ_{g0}, p_{g0} and φ_{g0} are independent of both t and $\boldsymbol{r}$. Equation (5.8) fixes the arbitrary constant in the potential of this Newtonianlike theory and it prevents the universe to be empty, i.e. ρ_{g0} may now be different from zero. With $\Lambda = 0$ one has only an empty universe as is obvious from equation (5.8).

5.3.2 Procedures

We again suppose here that all quantities are functions of $\chi = \omega t + \boldsymbol{k} \cdot \boldsymbol{r}$ alone (representing "one family"), where ω is again the angular frequency ($\omega = 2\pi\,\nu$), ν is the frequency and $\boldsymbol{k}$ is the wave vector. Hence it is possible to put equations (5.1) - (5.4) into the form

$$\omega \rho' + \boldsymbol{k} \cdot (\rho \boldsymbol{v})' = 0, \tag{5.10}$$

$$\rho\,(\omega + \boldsymbol{v} \cdot \boldsymbol{k})\,\boldsymbol{v}' = -\boldsymbol{k} p' - \rho \boldsymbol{k} \varphi', \tag{5.11}$$

$$k^2 \varphi'' + \Lambda \varphi = 4\pi G \rho, \tag{5.12}$$

$$p = K \rho^{\Gamma}. \tag{5.13}$$

As we did previously, we obtain from these equations:

$$(\omega + \boldsymbol{k} \cdot \boldsymbol{v})\rho = \in_g = \omega \rho_{g0} \tag{5.14}$$

and

$$\frac{\omega^2 \rho_{g0}^2 \rho'}{\rho^2} = k^2 K \Gamma \rho^{\Gamma-1} \rho' + \rho k^2 \varphi'. \tag{5.15}$$

Integrating (5.15) with respect to χ gives

$$\varphi = \frac{k_J^2 v_s^2}{\Lambda} + \frac{\omega^2 (1 - \eta^{-2})}{2k^2} + \frac{v_s^2 \left(1 - \eta^{\Gamma-1}\right)}{\Gamma - 1}, \tag{5.16}$$

where $v_s = \sqrt{K \Gamma \rho_{g0}^{\Gamma-1}}$ is the speed of sound in the medium, $k_J = 2\sqrt{\pi\, G\, \rho_{g0}/v_s^2}$ is the Jeans' wavenumber and $\eta = \rho/\rho_{g0}$ is the density normalized to the equilibrium density. The integration constant was determined by substituting zero order quantities into the equation obtained after integration.

One of our purposes, as in the case of previous chapters, is to see which one of the following three possible processes leads faster to the determination of the higher order coefficients.

Procedure 1: (Reduction to a nonlinear differential equation of second order)

We put equation (5.16) into equation (5.12) and rearrange to get

$$\left(k^2 v_s^2 \eta^{\Gamma+1} - \omega^2\right) \eta\eta'' + k_J^2 v_s^2 \eta^4 (\eta - 1) + \left(k^2 v_s^2 (\Gamma - 2) \eta^{\Gamma+1} + 3\omega^2\right) \eta'^2$$
$$+ \frac{\Lambda\eta^2}{k^2}\left[\frac{\omega^2\left(\eta^2 - 1\right)}{2} - \frac{k^2 v_s^2 \eta^2 \left(\eta^{\Gamma-1} - 1\right)}{(\Gamma - 1)}\right] = 0, \qquad (5.17)$$

which is a nonlinear ordinary differential equation of *second order*. We solve this using the Fourier analysis explained previously to determine the dispersion relation and the higher order coefficients of the perturbed density.

As Λ is extremely small we may approximate it by zero for practical purposes in these formulae (or even at once in equation (5.17)), simplifying them seriously. However, the introduction of Λ was essential in the approach in order to have a consistent basis.

Procedure 2: (Reduction to a nonlinear differential equation of first order)

Here we integrate equation (5.12) with respect to χ to have

$$k^2 \varphi' + \int \left(\Lambda\varphi - k_J^2 v_s^2 \eta\right) d\chi = 0. \qquad (5.18)$$

Substituting equation (5.16) into equation (5.18) and rearranging yields

$$\frac{\left(\omega^2 - k^2 v_s^2 \eta^{\Gamma+1}\right)\eta'}{\eta^3} + \Lambda\left[\chi\left(\frac{k_J^2 v_s^2}{\Lambda} + \frac{\omega^2}{2k^2} + \frac{v_s^2}{\Gamma - 1}\right)\right.$$
$$\left. - \int \left(\frac{\omega^2 \eta^{-2}}{2k^2} + \frac{v_s^2 \eta^{\Gamma-1}}{\Gamma - 1} + \frac{k_J^2 v_s^2 \eta}{\Lambda}\right) d\chi\right] = 0. \qquad (5.19)$$

This is a nonlinear ordinary differential equation of *first order* (again overlooking for the moment the integral sign). We use this, just as equation (5.17), in the determination of higher order coefficients in the density expression.

Procedure 3: (Reduction to a fully integrated equation)

Integrating equation (5.12) twice with respect to χ yields

$$k^2\left(\frac{k_J^2 v_s^2}{\Lambda} - \varphi\right) = \int \left[\int \left(\Lambda\varphi - k_J^2 v_s^2 \eta\right) d\chi\right] d\chi. \qquad (5.20)$$

We then substitute equation (5.16) into equation (5.20) to have

$$\frac{\omega^2\left(1-\eta^{-2}\right)}{2}+\frac{k^2 v_s^2\left(1-\eta^{\Gamma-1}\right)}{\Gamma-1}+\left(\frac{\Lambda\omega^2}{2k^2}+\frac{\Lambda v_s^2}{\Gamma-1}+k_J^2 v_s^2\right)\frac{\chi^2}{2}$$

$$-\Lambda\int\left[\int\left(\frac{\omega^2\eta^{-2}}{2k^2}+\frac{v_s^2\eta^{\Gamma-1}}{\Gamma-1}+\frac{k_J^2 v_s^2\eta}{\Lambda}\right)d\chi\right]d\chi=0. \qquad (5.21)$$

This is a *fully integrated* equation (again overlooking for the moment the integral signs) which can be used, just like equation (5.17) or equation (5.19), in the determination of the dispersion relation and the higher order coefficients of the perturbed density.

5.3.3 Dispersion relation and higher order coefficients

As stated in the beginning of the section 5.3.2, we restrict ourselves to one 'family' of terms all having the same phase velocity $j\omega/jk$, i.e. their argument is $j\chi$, with j an integer from 1 to infinity or N when considering a finite number of terms. Thus we write the normalized mass density as:

$$\eta=\sum_{j=0}^{N} a_{jg}A^j\exp\left(ij\chi\right), \qquad (5.22)$$

where a_{jg}'s (with N going to infinity) are the coefficients to be determined, $a_{0g}=1=a_{1g}$ and A is the initial amplitude of the first order perturbation (relative to the equilibrium density). Substituting equation (5.22) into either equation (5.17), equation (5.19) or equation (5.21) and linearizing one obtains

$$\boxed{\omega^2=k^2v_s^2\left(1-\frac{k_J^2}{k^2-\Lambda}\right).} \qquad (5.23)$$

Equation (5.23) gives the relationship between the angular frequency and wavenumber (together with equilibrium and basic constants). Moreover, the dispersion relation suggests the occurrence of instabilities when the wavenumber takes the values between $\sqrt{\Lambda}$ and $\sqrt{\Lambda+k_J^2}$ when $\Lambda>0$ and $k^2<k_J^2+\Lambda$ when $\Lambda<0$.

Ignoring the gravitational effects on fluctuation (i.e. allowing G (and hence k_J) in equation (5.23) to approach zero) one recovers the dispersion relation for sound waves. For a better insight see the graphical illustration in Figure 5.1 for $\Lambda>0$ and figure 5.2 for $\Lambda<0$.

Putting $\Lambda=0$ into equation (5.23), we obtain the dispersion relation of Jeans for gravitational waves in an infinite Newtonian gravitating medium

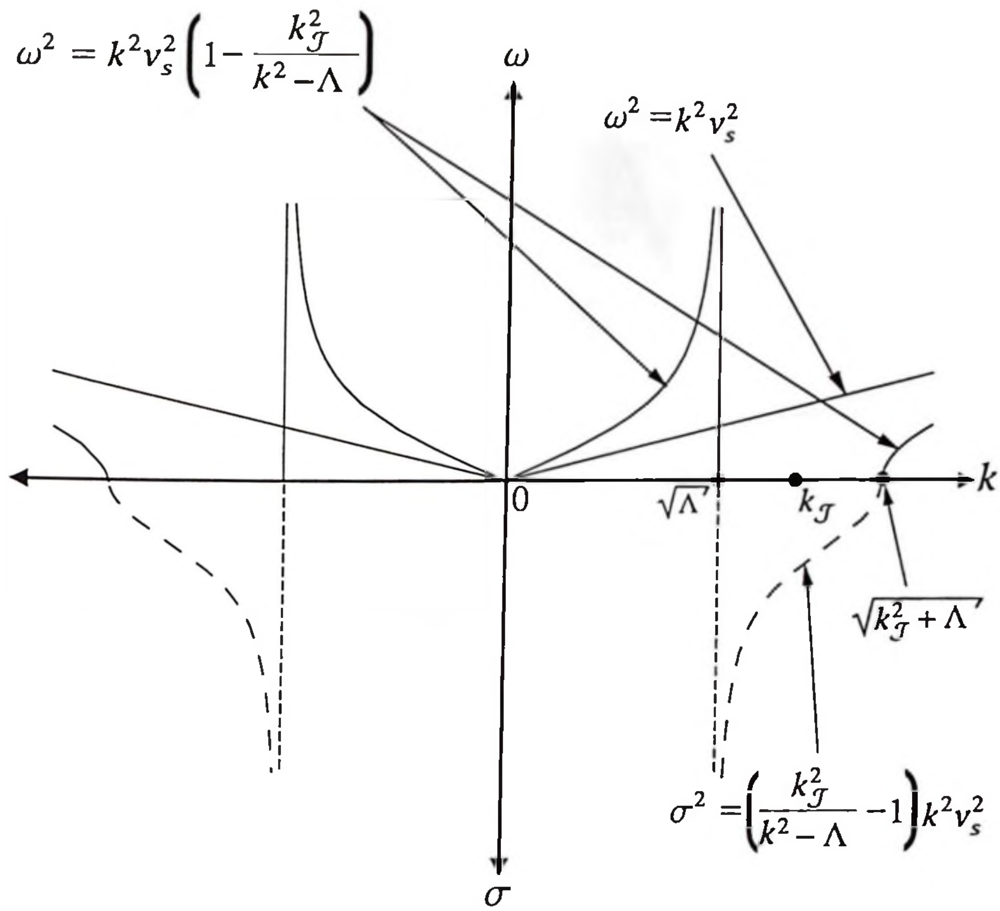

Figure 5.1: *The figure shows the $k - \omega$ plane being fused with the $k - \sigma$ plane, where k is the real wavenumber, ω is the real angular frequency and $\sigma = i\omega$ is the imaginary angular frequency corresponding to the growth rate of the instability. The negative parts of ω and σ were suppressed. On these planes we sketched, the graphs of the dispersion relations for the gravitational perturbations with a cosmological constant ($\Lambda > 0$), gravitational perturbations in the limit $\Lambda = 0$ and sound waves (the case where gravitational effects are ignored i.e. $G \to 0$). The Jeans's wavenumber is $k_J = \sqrt{4\pi G \rho_{g0}/v_s^2}$.*

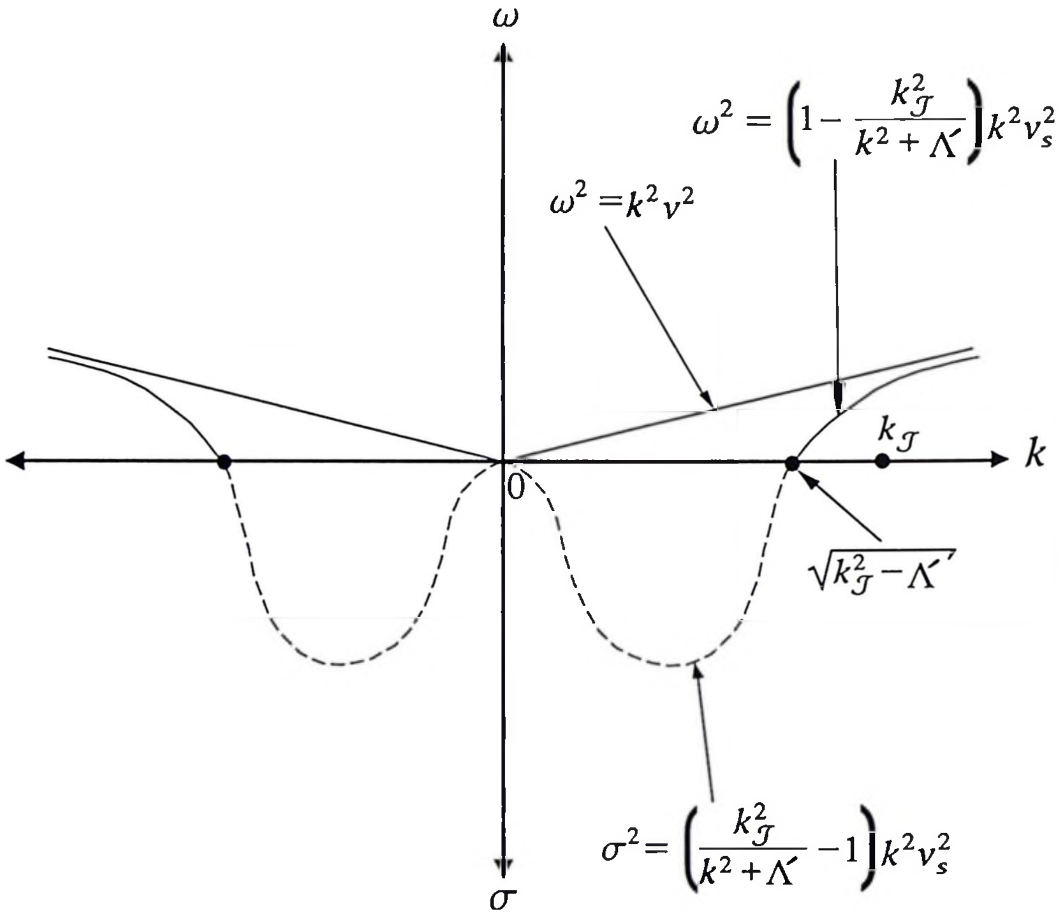

Figure 5.2: *The figure showing again the fusion of $k-\omega$ plane and $k-\sigma$ plane, where k is the real wavenumber, ω is the real angular frequency and $\sigma = i\omega$ is the imaginary angular frequency corresponding to the growth rate of the instability. The negative parts of ω and σ were suppressed. On these planes we sketched, the graphs of the dispersion relations for the gravitational perturbations with a cosmological constant ($\Lambda < 0$, where $\Lambda = -\Lambda'$ with Λ' being a positive quantity) and sound waves (the case where gravitational effects are ignored i.e. $G \to 0$). The Jeans's wavenumber is $k_J = \sqrt{4\pi G \rho_{g0}/v_s^2}$. Extrema are for $k = 0$ ($\sigma = 0$) and $k^2 = k_J\sqrt{\Lambda'} - \Lambda'$ $\left(\sigma_{\max} = \left(k_J - \sqrt{\Lambda'}\right) v_s\right)$.*

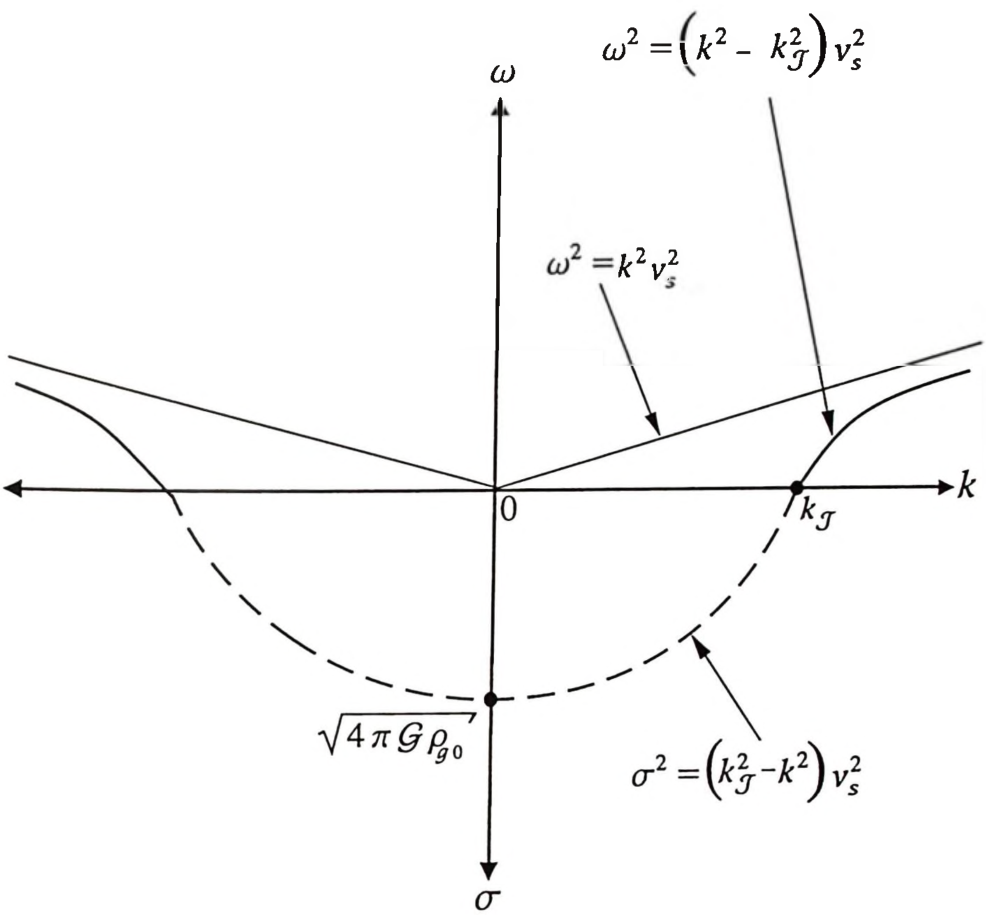

Figure 5.3: *The figure shows the $k - \omega$ plane being fused with the $k - \sigma$ plane, where k is the real wavenumber, ω is the real angular frequency and $\sigma = i\omega$ is the imaginary angular frequency corresponding to the growth rate of the instability. The negative parts of ω and σ were suppressed. On these planes we sketched, the graphs of the dispersion relations for the gravitational perturbations in the limit $\Lambda = 0$ and sound waves (the case where gravitational effects are ignored i.e. $G \to 0$). We see from the graph that instability starts below $k_J = \sqrt{4\pi G \rho_{g0}/v_s^2}$. We also note that the maximum growth rate is $\sigma_{\max} = \sqrt{4\pi G \rho_{g0}}$.*

(see figure 5.3). Indeed the dispersion relation of Jeans and his stability criterion are subject to heavy criticism as the equilibrium from which they are derived is impossible. However, introducing Λ allows a correct equilibrium and thus a correct dispersion relation. Letting $\Lambda \to 0$ corresponds to approaching the Newtonian case. Thus the analysis, with $\Lambda \to 0$, confirms the Jeans's criterion given in [58, pp. 588-589], at least for the k-values where Λ has barely an influence.

Some comments on the case when $\Lambda = 0$

1. The sound speed $v_s = \sqrt{\Gamma p_{g0}/\rho_{g0}} = \sqrt{\Gamma k_B T_0/m}$ varies proportionally to $\sqrt{\Gamma}$: the less compressible the fluid is, the larger the speed. The smaller the mass m of the particles, the higher the speed. The larger the temperature T_0, the higher the speed. (Note that the number density is irrelevant unless going to refinements). The sound speed is phase velocity as well as group velocity.

2. In the gravitational case the phase velocity is

$$\frac{\omega}{k} = v_s\sqrt{1 - \frac{k_J^2}{k^2}} \equiv v_{ph} \qquad (k > k_J)$$

which is smaller than the sound velocity when neglecting the gravitation. The group velocity is

$$\frac{\partial \omega}{\partial k} = v_s\left(1 - \frac{k_J^2}{k^2}\right)^{-\frac{1}{2}} \equiv v_g \qquad (k > k_J)$$

so that $v_{ph}v_g = v_s^2$. This is similar to several electromagnetic situations (e.g. plasma case, light in a refracting medium or micro waves in a travelling wave tube). However in the electromagnetic cases one has usually that $v_{ph} \geq c$ and $v_g \leq c$ as certainly has to be when c is the light speed in vacuum.

3. Jeans' wave length is

$$\lambda_J = \frac{2\pi}{k_J} = 2\pi\sqrt{\frac{\Gamma p_{g0}}{4\pi G \rho_{g0}^2}} = \sqrt{\frac{\pi \Gamma k_B T_0}{m^2 G n_{g0}}}$$

which decreases with increasing number density and increasing with increasing temperature.

4. Jeans' mass is

$$M_J = \rho_{g0}\,\lambda_J^3 = \frac{1}{\sqrt{\rho_{g0}}}\left(\frac{\pi\Gamma k_B T_0}{mG}\right)^{\frac{3}{2}}.$$

Hence the larger the temperature the larger the λ_J and the larger the M_J, similarly the larger the Γ the larger the M_J: the greater the pressure, the more resistance against the gravitation and thus more mass is needed. However, the larger the mass density, the smaller the M_J.

This looks surprising at first sight, but is quite understandable as a large mass density yields a stronger gravitation which can overcome the pressure at a shorter distance. Hence stars born out of the interstellar medium have a large mass, but galaxies born out of the much more tenuous intergalactic medium have still much larger masses.

Higher order coefficients

We used equations (5.17), (5.19) and (5.21) to determined (with the assistance of *mathematica* program) the coefficients of the higher order terms in the expressions of density. We give in appendix B.5.1 the expressions for the first ten coefficients. Making appropriate interchanges in these results one gets back some of the coefficients obtained previously in [46]. For example one may recover results for the case of plasma with electron pressure, if one replaces Λ and ω_β $\left(\equiv (1+\Gamma)\,(k^2-\Lambda)\,/k_J^2 \right)$ by zero and $-\omega_\Omega$ $\left(\equiv -(1+\Gamma)\; k^2 v_{s-}^2/\omega_-^2 \right)$ respectively. The coefficients for the one component cold plasma (see table 2.1) on the other hand, may be recovered by replacing both Λ and ω_β by zero.

For the cases of velocity and potential, where we used respectively equations (5.14) and (5.16), we determined (again with the assistance of the appropriate *mathematica* program) the coefficients up to order ten. We summarize these results in appendices B.5.2 and B.5.3 respectively.

5.3.4 Convergence

The radius of convergence (by which we mean here the maximum amplitude of the first order term allowing convergence) can be estimated by using the graphical method which was elaborated thoroughly in chapter 2. In Table 5.1 we give the summary of the obtained maximum values of $|A|$ (radius of convergence) for some values of the parameter ω_β with Λ and Γ being put equal to 0 and 5/3 (corresponding to adiabatic perturbations of a monoatomic gas, i.e. $\gamma = 5/3$) respectively.

From the table we see that the maximum value of $|A|$ becomes e^{-1} (which may also be determined analytically; cf. section 2.3.1.1) when $\omega_\beta = 0$. We then notice that when $\omega_\beta \neq 0$, the radius of convergence increases with the increasing value of ω_β. This increase continues till when the decreasing wavelength reaches the value of the wavelength of Jeans. In this case the dispersion relation curve in the (k, ω) diagram moves away from the sound waves dispersion relation curve (cf. figure 5.3).

The convergence is almost optimal for $k = k_J$, where the dispersion relation curve approach a vertical in the (k, ω) diagram and thus the phase

ω_β	k_J/k	$\|A\|$
0	∞	$< 0.367 \approx e^{-1}$
1/2	$4/\sqrt{3}$	$< 0.22 \approx 0.5980\ e^{-1}$
1	$\left(2\sqrt{2}\right)/\sqrt{3}$	$< 0.25 \approx 0.6796\ e^{-1}$
2	$2/\sqrt{3}$	$< 0.3 \approx 0.8155\ e^{-1}$
8/3	1	$< 0.98 \approx 2.6639\ e^{-1}$
3	$\left(2\sqrt{2}\right)/3$	$< 0.2 \approx 0.5437\ e^{-1}$
8	$1/\sqrt{3}$	$< 0.02 \approx 0.0544\ e^{-1}$

Table 5.1: *The graphically estimated values of the radius of convergence (i.e. $|A|$) for some values of ω_β, where $\Lambda = 0$ and $\Gamma = 5/3$, k is the wave number, $k_J = 2\sqrt{\pi G \rho_{g0}/v_s^2}$ is the Jeans' wavenumber, $v_s = \sqrt{K\Gamma\rho_{g0}^{\Gamma-1}}$ is the speed of sound and $\omega_\beta = (1+\Gamma)\left(k^2 - \Lambda\right)/k_J^2$. The values under $|A|$ are based on the terms up to tenth order.*

velocity varies strongly with k. This is the opposite of the cases of sound waves or ion acoustic waves, where the phase velocity is independent of k and yields zero convergence. In fact as $A \approx 1$ corresponds to a wave with almost full density of the medium as amplitude, it is impossible to achieve better. It practically means that the higher orders are very small.

The radius of convergence starts then to decrease when the wavelength decreases more and more below the wavelength of Jeans. This implies that when the gravitational effect becomes less important and as we approach sound waves the convergence becomes worse and worse.

5.4 Conclusion

The addition of the cosmological tern in the Newtonian field equation allows the homogeneous gravitational equilibrium to be possible for an infinite medium. For the limit $\Lambda \to 0$ we obtain Jeans's dispersion relation and Jeans's criterion, giving them now a much stronger basis. Some of the results obtained in this work were compared to the results determined previously for the plasma case in [46]. The methods used are those elaborated in chapter 3. Some of the results determined in this chapter will appear in [59].

Clearly the higher order terms allow corrections to the linearized theory and yield more insight especially concerning the convergence.

We may draw quite suggestive conclusions from our analysis, but as it is unclear how close the basis of the theory with Λ is as an approximation for reality we may not tell how close the results correspond to reality when looking for cosmological application. Nevertheless they may give a close hint, and certainly form a guideline when the Einsteinian equations are used. Moreover the expansion of the universe has to be taken into account as well. By all means the introduction of the cosmological term removes the fundamental inconsistency in the equilibrium. However the consolidation of Jeans's criterion in view of application to stars, galaxies and clusters of galaxies is a nice result, especially as this consolidation comes in two ways: (a) the consistent basic equations and equilibrium (b) the higher order terms. In relation with the second effect it may be important to mention that the higher orders might have indicated a very bad convergence (we experienced the example of plain sound waves), while here the convergence is extremely good near the critical wavelength.

Chapter 6

Nonuniform gravitational equilibrium and its stability

6.1 Introduction

As in chapter 5, we again consider the medium infinite in all space dimensions and obeying the Newtonian law of gravitation to which a cosmological term is added. But now, as opposed to the analysis in chapter 5, we analyze the equilibria involving a varying gravitational potential. As we remarked previously, the interest in the cosmological term has been revived in particular due to the recent findings of the supernovae acting as standard candles to look back into the distant past of the universe.

The basic system of equations to be considered here are still the same equations of continuity, motion, Poisson with cosmological term and polytropics given by respectively equations (5.1) - (5.4). Putting zero order quantities, with $\boldsymbol{v}_0 = 0$, (the extension to a dynamic universe, $\boldsymbol{v}_0 \neq 0$, will be considered later) into these equations we have

$$\partial_t \rho_0 = 0, \tag{6.1}$$

$$\boldsymbol{\nabla} p_0 = -\rho_0 \boldsymbol{\nabla} \varphi_0, \tag{6.2}$$

$$\Delta\varphi_0 + \Lambda\varphi_0 = 4\pi G\rho_0, \tag{6.3}$$

$$p_0 = K\rho_0^{\Gamma}. \tag{6.4}$$

Here we note that ρ_0 and hence p_0 and φ_0 are independent of time t but dependent on space, $\boldsymbol{r}$. In order to differentiate them from the previously seen quantities, we respectively denote them often by: $\rho_0(\boldsymbol{r})$, $p_0(\boldsymbol{r})$ and $\varphi_0(\boldsymbol{r})$. We recall that Λ is the cosmological constant and very small. With these notations and equations (6.1) - (6.4), we analyze the following cases:

6.2 Analysis for the isothermal case ($\Gamma = 1$)

It is well known that the pressure at equilibrium can also be expressed as $p_0(r) = \rho_0(r)k_B T_0/m$ with m the (average) molecular weight. Using this in (6.4) with $\Gamma = 1$ (the isothermal case), one gets

$$K = \frac{k_B T_0}{m}. \tag{6.5}$$

Again putting (6.4) into (6.2) produces $K\nabla\rho_0(r) = -\rho_0(r)\nabla\varphi_0(r) \Rightarrow \nabla(K \ln \rho_0(r) + \varphi_0(r) = 0) \Rightarrow \ln\rho_0(r) + (\varphi_0(r)/K) = \ln\rho_{00}$ from which we get

$$\boxed{\rho_0(r) = \rho_{00}\, e^{-\frac{\varphi_0(r)}{K}} = \rho_{00}\, e^{-\frac{m\,\varphi_0(r)}{k_B T_0}}} \tag{6.6}$$

(cf. The Boltzmann factor, the barometric formula, ...). Note: $\rho_0(r)$ is the equilibrium density (space dependent), ρ_{00} is the density (a constant) at the place where $\varphi_0(r) = 0$. This formula (equation(6.6)) is explicitly independent of Λ; however, $\varphi_0(r)$ depends on Λ in view of the gravitational field equation

$$\boxed{\Delta\varphi_0(r) + \Lambda\varphi_0(r) = 4\pi G\rho_0(r) = 4\pi G\rho_{00}\, e^{-\frac{m\,\varphi_0(r)}{k_B T_0}},} \tag{6.7}$$

which determines $\varphi_0(r)$, which hence depends on Λ. The equation is similar to the one fixing the Debye potential and the Debye length, but generalizes it because of the term containing Λ. Thanks to this term it makes sense to consider the concepts of a gravitational Debye potential and Debye length on a safer basis than just the devious similarity with the plasma situation. That this is possible now is due to the cosmological term which in a sense corresponds to a 'negative' mass if $\Lambda\,\varphi_0(r) > 0$.

For $|m\varphi_0(r)| << k_B T_0$, the exponential in (6.7) can be expanded and linearized to give

$$\Delta\varphi_0(r) + \left(\Lambda + \frac{4\,\pi\, m\, G\,\rho_{00}}{k_B\, T_0}\right)\varphi_0(r) = 4\pi G\rho_{00}. \tag{6.8}$$

Putting $\lambda_{DG}^{-2} = \Lambda + (4\,\pi\, m\, G\,\rho_{00})\,/\,(k_B\, T_0)$ then (6.8) becomes

$$\Delta\varphi_0(r) + \lambda_{DG}^{-2}\,\varphi_0(r) = 4\pi G\rho_{00}, \tag{6.9}$$

where λ_{DG}^2 is called the gravitational Debye length. Its customary definition by $\lambda_{DG}^2 = (k_B\, T_0)\,/\,(4\,\pi\, m\, G\,\rho_{00})$, is usually adequate as Λ is assumed to be extremely small with respect to the other term. The one dimensional solution to equation (6.9) is

$$\varphi_0(x) = 4\,\pi\, G\,\rho_{00}\,\lambda_{DG}^2 + \varphi_{00}\cos\left[(x/\lambda_{DG}) + \Psi\right] \tag{6.10}$$

in Cartesian coordinates or

$$\varphi_0(r) = \frac{\varphi_{00} \cos\left[(r/\lambda_{DG}) + \Psi\right]}{r} + 4\,\pi\,G\,\rho_{00}\,\lambda_{DG}^2 \tag{6.11}$$

in spherical coordinates. However, those solutions involve some inconsistency with the condition $|m\varphi_0(r)| << k_B T_0$. Indeed the last term of the expression (6.10) or (6.11) may be rewritten as $k_B T_0/m$ and the approximation is only valuable in the regions where the cosinusoidal term in (6.10) or (6.11) provides sufficient compensation.

6.3 Analysis for the other cases i.e. $\Gamma \neq 1$

Substituting (6.4) into (6.2) we get:

$$\nabla K\rho_0^\Gamma(r) = -\rho_0(r)\nabla\varphi_0(r) \Rightarrow K\Gamma\rho_0^{\Gamma-1}(r)\nabla\rho_0(r) = -\rho_0(r)\nabla\varphi_0(r)$$

$$\Rightarrow \frac{K\Gamma}{\Gamma-1}\nabla\rho_0^{\Gamma-1}(r) = -\nabla\varphi_0(r) \Rightarrow \varphi_0(r) = -\frac{K\Gamma}{\Gamma-1}\rho_0^{\Gamma-1}(r) + \varphi_{00},$$

where φ_0 is a kind of cosmological background potential. Substituting this into the field equation (6.3), we get

$$\boxed{-\tfrac{K\Gamma}{\Gamma-1}\Delta\rho_0^{\Gamma-1}(r) - \tfrac{K\Gamma\Lambda}{\Gamma-1}\rho_0^{\Gamma-1}(r) + \Lambda\varphi_{00} = 4\pi G\rho_0(r).} \tag{6.12}$$

This constitutes to a generalization of the Lane - Emden equation, famous in the early days of stellar structure (cf. Chandrasekhar [58]), for which $\Lambda = 0$.

It is a nonlinear partial differential equation. In the case of spherical symmetry with $\Lambda = 0$ (stellar case), one has simple closed solutions for the following values of Γ: ∞ (incompressible), 2 and 6/5. Moreover, for $\Lambda = 0$, there is, for any Γ ($\neq 1$) and for spherical, cylindrical or Cartesian coordinates, the so-called singular solution of the type Ar^p, with A and p fixed in terms of Γ (if $p < 0$ the solution is rejected because it is singular at the center of the star, if $p > 0$ the solution is rejected as it yields zero density in its center).

6.3.1 A simple example: $\Gamma = 2$

If $\Gamma = 2$ then (6.12) becomes linear

$$\Delta\rho_0(r) + \left(\Lambda + \frac{2\pi G}{K}\right)\rho_0(r) = \frac{\Lambda\varphi_{00}}{2K}. \tag{6.13}$$

Replacing $\rho_0(r)$ by

$$\rho_0(\boldsymbol{r}) = a_0 + \sum_{j=1}^{N} (a_j \cos j\boldsymbol{k}\cdot\boldsymbol{r} + b_j \sin j\boldsymbol{k}\cdot\boldsymbol{r}), \tag{6.14}$$

gives

$$a_0 = \frac{\Lambda\varphi_{00}}{2(K\Lambda + 2\pi G)}, \tag{6.15}$$

$$2Kj^2k^2a_j - 2K\Lambda a_j = 4\pi G a_j \quad \Rightarrow [2K(k^2j^2 - \Lambda) - 4\pi G]a_j = 0 \tag{6.16}$$

and

$$[2K(k^2j^2 - \Lambda) - 4\pi G]b_j = 0. \tag{6.17}$$

Hence

$(kj)^2 = (2\pi G + K\Lambda)/K$, all other $a_j = 0$. Similarly for b_j in (6.17).

With suitable choice of the origin:

$$\rho_0(\boldsymbol{r}) = B + a\cos(k_0\boldsymbol{r}) \tag{6.18}$$

where

$$B = \frac{\Lambda\varphi_{00}}{2(K\Lambda + 2\pi G)} \tag{6.19}$$

and

$$k_0 = \sqrt{\frac{2\pi G}{K} + \Lambda}. \tag{6.20}$$

As $\rho_0(\boldsymbol{r}) \geq 0$,

$$\varphi_{00}\,\Lambda \geq 2\,(K\Lambda + 2\pi G)\,|a| \tag{6.21}$$

Note that this implies $\varphi_{00}\,\Lambda > 0$: if Λ is zero then $B = 0$ and $a = 0$, which reduces the situation to an empty universe. It is remarkable that even in a nonhomogeneous universe the cosmological constant is required. Similarly φ_{00}, although arbitrary, may not simply be put equal to zero if we want a varying density. Notice that (6.18) implies as well a more general form with a sum of cosines:

$$\boxed{\rho_0(\boldsymbol{r}) = B + a\cos k_0x + b\cos k_0y + c\cos k_0z} \tag{6.22}$$

with a, b and c arbitrary but restricted by the condition $\rho_0(\boldsymbol{r}) \geq 0$ or

$$B \geq |a| + |b| + |c|. \tag{6.23}$$

The other equilibrium quantities are given by

$$p_0(\boldsymbol{r}) = K\rho_0^2(\boldsymbol{r}) \tag{6.24}$$

$$\varphi_0(\boldsymbol{r}) = -2K\rho_0(\boldsymbol{r}) + \varphi_{00}. \tag{6.25}$$

6.3.1.1 Remark:

If $K\Lambda + 2\pi G = 0$ (or $B = \infty$, see equation (6.19)), which seems physically unrealistic in view of the supposed extreme smallness of Λ, then (6.13) has a solution of the type

$$\rho_0(\boldsymbol{r}) = ax^2 + by^2 + cz^2 + dx + ey + fz + g \tag{6.26}$$

with a, b, c, $\cdots$, g arbitrary constants, except for the condition $\rho_0(\boldsymbol{r}) \geq 0$. As x, y and z may have arbitrary positive and negative values in an infinite universe this condition requires $d = e = f = 0$ and a, b, c, ≥ 0. However this would yield infinite density for $|x|$, $|y|$ or $|z| \to \infty$ requiring $a = b = c = 0$ which leads back to a homogeneous density.

6.3.1.2 Discussion of equation (6.22)

It is remarkable that the amplitudes in (6.22) are not fixed: with any of them satisfying condition (6.23), corresponds a possible equilibrium. This suggests the idea that a certain equilibrium may evolve smoothly to another equilibrium through a continuous series of equilibria, not really by instability, but rather like a ball which is in a horizontal gutter or on a horizontal plane: it may be at any place. However, in the gutter there are still transversal oscillations possible, and if the gutter is inversed to be a ridge there are transversal instabilities. For the ball on the horizontal plane we speak of an indifferent equilibrium in all directions, for all perturbations.

The kind of equilibrium of the form (6.22) is quite interesting in view of the so-called tessellation of the universe. Indeed during the last decennium it was observed that there were a kind of accumulating "walls" in the universe, where the density of the galaxies is higher than inside the regions surrounded by those 'walls' forming irregular polyhedra (of several million light-year across).

The physical interpretation of the tessellation is based on the interpretation of Jeans' instability. In the customary view of it an accumulation of matter in a gravitating medium will increase provided it is sufficiently large (depending on pressure, etc.). However, if the medium has a depletion of matter at a certain place and of sufficient magnitude, this depletion will be enhanced by the same mechanism of Jeans' instability and its surroundings, its sides, (its "walls"), will increase their density. Hence the tessellation of the universe seems a natural phenomenon and the strengthening of this tessellation a quite natural evolution.

Actually equation (6.22) indicates such a kind of tessellation and the fact that the amplitudes are arbitrary suggest the possibility of an (easy) enhancement of the tessellation structure. This makes the investigation of the stability of the inhomogeneous universe corresponding to (6.22) very interesting.

6.3.1.3 Stability analysis

Perturbing and linearizing the relevant equations (5.1) - (5.4) yields

$$\partial_t \rho_1 + div(\rho_0 \boldsymbol{v}_1) = 0 \tag{6.27}$$

$$\rho_0 \partial_t \boldsymbol{v}_1 = -\nabla p_1 - \rho_0 \nabla \varphi_1 - \rho_1 \nabla \varphi_0(\boldsymbol{r}) \tag{6.28}$$

$$\Delta \varphi_1 + \Lambda \varphi_1 = 4\pi G \rho_1 \tag{6.29}$$

$$p_1 = 2K \rho_0 \rho_1 \tag{6.30}$$

where we recall that ρ_0, p_0 and φ_0 are not constants but given by equations (6.19), (6.20), (6.22) - (6.25). Eliminating v_1 and p_1 yields

$$\partial^2_{tt} \rho_1 = 2K\Delta(\rho_0 \rho_1) + \rho_0 \Delta \varphi_1 + \rho_1 \Delta \varphi_0(\boldsymbol{r}) + \nabla \rho_0 \cdot \nabla \varphi_1 + \nabla \rho_1 \cdot \nabla \varphi_0(\boldsymbol{r})$$

$$= 2K\,(\rho_0 \Delta \rho_1 + \rho_1 \Delta \rho_0 + 2\nabla \rho_0 \cdot \nabla \rho_1) + \rho_0 \Delta \varphi_1 + \rho_1 \Delta \varphi_0(\boldsymbol{r}) + \nabla \rho_0 \cdot \nabla \varphi_1 + \nabla \rho_1 \cdot \nabla \varphi_0(\boldsymbol{r}).$$

Using the expressions for ρ_0 and φ_0 we may simplify this:

$$\partial^2_{tt} \rho_1 = 2K \rho_0 \Delta \rho_1 + 2K \nabla \rho_0 \cdot \nabla \rho_1 + \rho_0 \Delta \varphi_1 + \nabla \rho_0 \cdot \nabla \varphi_1. \tag{6.31}$$

The elimination of φ_1 using (6.29) is not simple especially in view of the cosmological term. It is simple to eliminate ρ_1, yielding a linear partial differential equation of fourth order in φ, however with non-constant coefficients. Hence, we rather try first to handle (6.29) by taking a Fourier series or integral for ρ_1 and restricting ourselves to the x-dependence only for the sake of convenience. Actually the problem is linear in the perturbed quantities, however, bilinear in equilibrium and perturbed quantities.

Thus we consider the sum or integral:

$$\rho_1 = \sum_{j=0}^{\infty} a_j e^{\sigma_j t} \cos(jx + \psi_j) \tag{6.32}$$

where we have included the time dependence explicitly: we have taken it as exponential in view of equation (6.31): Equations (6.29) and (6.32) yield,

$$\varphi_1 = 4\pi G \sum_{j=0}^{\infty} \frac{a_j\, e^{\sigma_j t}}{\Lambda - j^2} \cos(jx + \psi_j) + b\, e^{\sigma t} \cos(\sqrt{\Lambda} x + \psi).$$

Here a_j, ψ_j, σ_j, b, ψ and σ are still arbitrary constants. σ_j and σ may still be complex. Note that if $\sqrt{\Lambda}$ coincides with a particular j we have 'absorption' in equation (6.29) and the corresponding solution is $bxe^{\sigma t}\cos(\sqrt{\Lambda}x + \psi)$. However we do not expect this as an appropriate perturbation in an infinite universe.

As the equation (6.31) is linear in ρ_1 and φ_1 we may work with a single term of ρ_1:

$$\sigma_j^2\, a_j \cos(jx+\psi_j) = -\left(2K + \frac{4\pi G}{\Lambda - j^2}\right) j^2\, a_j\, \rho_0(r) \cos(jx+\psi_j) + \left(2K + \frac{4\pi G}{\Lambda - j^2}\right)$$

$$\times k_0\, j\, a\, a_j\, \sin k_0 x \sin(jx+\psi_j) - b\Lambda\rho_0 \cos(\sqrt{\Lambda}x+\psi) + a\, b\, k_0\, \sqrt{\Lambda} \sin k_0 x \sin(\sqrt{\Lambda}x+\psi)$$

where we have dropped the time factor. Replacing $2K$ by $4\pi G/(k_0^2 - \Lambda)$ and expliciting ρ_0 yields

$$\sigma_j^2\, a_j \cos(jx + \psi_j) = -4\pi G\, j\, a_j\, a \left(\frac{1}{k_0^2 - \Lambda} + \frac{1}{\Lambda - j^2}\right) \Big[j \cos k_0 x \cos(jx + \psi_j)$$

$$-k_0 \sin k_0 x \sin(jx+\psi_j) + jB\cos(jx+\psi_j) \Big] - a\, b\, \sqrt{\Lambda} \Big[\sqrt{\Lambda} \cos k_0 x \cos(\sqrt{\Lambda}\, x+\psi)$$

$$-k_0 \sin k_0 x \sin(\sqrt{\Lambda}\, x + \psi) + \sqrt{\Lambda}\, B \cos(\sqrt{\Lambda} x + \psi) \Big]. \tag{6.33}$$

As $j \neq \pm\sqrt{\Lambda}$ there is no match between terms on the right hand side. The terms with products of (co)sines can not match the left hand side and have to vanish.

1. Take $b \neq 0$. For $k_0 = \pm\sqrt{\Lambda}$ and a suitable choice of ψ.

$$\sqrt{\Lambda} \cos k_0\, x \cos(\sqrt{\Lambda}\, x + \psi) - k_0 \sin k_0\, x \sin(\sqrt{\Lambda}\, x + \psi)$$

becomes a constant which cannot cancel with another term ($k_0 = \pm\sqrt{\Lambda} \neq j$). However with another choice of ψ the expression vanishes. However, the last term in (6.33) cannot match another term as $j \neq \pm\sqrt{\Lambda}$. Hence $B = 0$. which kills the similar term on the right hand side too and there is no match left for the left hand side.

2. Take $b = 0$. Then

$$j \cos k_0 x \cos(jx + \psi_j) - k_0 \sin k_0 x \sin(jx + \psi_j)$$

has still to disappear. This expression may vanish if $k_0 = \pm j$ but then its coefficient vanishes too, reducing the right hand side to zero. If the previous expression does not vanish, its coefficient has to vanish (i.e. $a_j = 0$ or $j^2 = k^2$) and this requires the left hand side to vanish, i.e. $\sigma_j = 0$ or $a_j = 0$

Finally only $a_j = a_{k_0}$ with $\sigma_j = \sigma_{k_0} = 0$ remains and we obtain with adapted notation

$$\rho_1 = a_{k_0} \cos(k_0 x + \psi_{k_0}) \tag{6.34}$$

$$\varphi_1 = \frac{4\pi G\, a_{k_0}}{\Lambda - k_0^2} \cos(k_0 x + \psi_{k_0}). \tag{6.35}$$

Hence we have marginal stability and moreover, in view of the arbitrariness of a_{k_0} and ψ_{k_0} the inhomogeneity of ρ may be enhanced or diminished by this perturbation. All this suits very well with the conjecture of indifferent equilibrium above.

6.3.2 Analysis based on the use of $\chi = \omega t + \boldsymbol{k} \cdot \boldsymbol{r}$ as argument

Due to the fact that we are considering the case where χ is now taken as an argument, then we think it is appropriate to make use of equations (5.10) - (5.14) with $\Gamma = 2$. However, putting now $\boldsymbol{v} = 0$ (more general $\boldsymbol{k} \cdot \boldsymbol{v} = 0$) to fix the constant $\in_g$, leads to a surprise as ρ_0 is not a constant in a nonuniform medium. Thus $\in_g$ and ω have to be zero. It follows that the left hand side of (5.11) vanishes too and the system is reduced at once to its equilibrium equations. This confirms our previous analysis . Starting from an equilibrium (or more general a situation with $\boldsymbol{k} \cdot \boldsymbol{v} = 0$) leads to neither an oscillation nor an instability. The equilibrium (or motion) is indifferent to motions based on a Fourier analysis. This result of the preceding section is now generalized to cases with motion for which $\boldsymbol{k} \cdot \boldsymbol{v} = 0$. Further generalization considers

$\omega + \boldsymbol{k} \cdot \boldsymbol{v} = 0$. But this requires $v_{\|} = -\omega/k$ to be constant which is just an irrelevant parallel displacement and then the situation is the same as $\omega = 0$.

The only motion which is allowed is then with $\boldsymbol{k} \cdot \boldsymbol{v} = 0$, restricting $\boldsymbol{v}$ to be one-dimensional (but of varying magnitude and sign) or possibly two dimensional. If the medium has a density varying like $\cos kx$ the amplitude of this may increase or decrease as this involves only motions perpendicular to x, in case $\boldsymbol{k}$, as exemplified in the previous section.

This cuts short at once our higher order analysis when starting from an inhomogeneous equilibrium. Further investigations should not use the hypothesis concerning χ, but rather a nonperiodic consideration, maybe just using a series development in t.

Note:

1. The above situation applies to most inhomogeneous media as the continuity equation is of general validity and similarly are the inertia terms in the equation of motion.
2. Still further generalization considers $(\omega + \boldsymbol{k} \cdot \boldsymbol{v})\rho = \in$.
3. Then the nonlinear Fourier analysis using χ is still useful, see e.g chapter 7 (the chapter on dynamic inhomogeneous universe).

6.3.3 General Γ

6.3.3.1 Power law for ρ_0

Consider equation (6.12) and let $\rho_0 = Ax^n$

$$-\frac{K\Gamma}{\Gamma-1}A^{\Gamma-1}(n(\Gamma-1))(n(\Gamma-1)-1)x^{(\Gamma-1)n-2}-\frac{K\Gamma\Lambda}{\Gamma-1}A^{\Gamma-1}x^{(\Gamma-1)n}+\Lambda\varphi_{00} = 4\pi G A x^n.$$

Equating some exponents yields

1. $(\Gamma-1)n-2=0$ and $(\Gamma-1)n = n \Rightarrow \Gamma = 2, n = 2$

   Moreover, equating coefficients:

   $-2KA2 = \Lambda\varphi_{00} \Rightarrow A = \frac{-\Lambda\varphi_{00}}{4K}$ and $-2KA\Lambda = 4\pi G A \Rightarrow -K\Lambda = 2\pi G$.

   Compare with the previous section. Anyway this solution is not suited for an infinite medium.

2. $(\Gamma-1)n-2 = n$ and $(\Gamma-1)n = 0 \Rightarrow \Gamma = 1$ or $n = 0$.

   $n = 0$ is not compatible with first equation (even without looking for the coefficients).

   For $\Gamma = 1$ we have the isothermal case corresponding to a different equation: see section 6.2.

6.3.3.2 Fourier series

Consider a Fourier series for the density $\rho_0(\boldsymbol{r})$:

$$\rho_0(\boldsymbol{r}) = \rho_{00} \sum_{j=0}^{\infty} \eta_j \, e^{ijh} \tag{6.36}$$

using the abbreviation $h = \boldsymbol{k}_g \cdot \boldsymbol{r}$ and with $\eta_0 = 1$. This is required to avoid negative density. Note that one may see ρ_{00} as the average density too. Using the binomial theorem one obtains

$$\begin{aligned}\frac{\rho_0(\boldsymbol{r})}{\rho_{00}} &= 1 + (\Gamma-1)\eta_1 e^{ih} + (\Gamma-1)\left[\eta_2 + \frac{(\Gamma-2)}{2}\eta_1^2\right] e^{2ih} \\ &+(\Gamma-1)\left[\eta_3 + (\Gamma-2)\eta_1\eta_2 + \frac{(\Gamma-2)(\Gamma-3)}{6}\eta_1^3\right] e^{3ih} \\ &+(\Gamma-1)\left[\eta_4 + \frac{(\Gamma-2)}{2}\left(\eta_2^2 + 2\eta_1\eta_3\right) + \frac{(\Gamma-2)(\Gamma-3)}{2}\eta_1^2\eta_2 \right. \\ &\left. + \frac{(\Gamma-2)(\Gamma-3)(\Gamma-4)}{24}\eta_1^4\right] e^{4ih} + \cdots\end{aligned} \tag{6.37}$$

Substitution in (6.12) yields

Zero order

$$-\frac{K\Gamma\Lambda}{\Gamma-1}\rho_{00}^{\Gamma-1} + \Lambda\varphi_{00} = 4\pi G\rho_{00} \tag{6.38}$$

as $\rho_{00} \neq 0$ we must have $\Lambda\varphi_{00} \neq 0$. As Λ is extremely small and φ_{00} (which plays the role of the potential of the homogeneous universe) is not small we may neglect the first term in (6.38) to a very good approximation. Hence

$$\boxed{\rho_{00} = \tfrac{\Lambda\varphi_{00}}{4\pi G}} \tag{6.39}$$

This is independent of Γ (Numerical verification that approximation is extremely good: with $\rho_{00} = 10^{-27}\text{kg/m}^3$, $G = 6.6\ 10^{-11}\text{m}^3/\text{s}^2\text{kg}$, $\varphi_{00} \approx c^2 = 9\ 10^{16}\text{m}^2/\text{s}^2$ one obtains $\Lambda \approx 10^{-53}\text{m}^{-2}$ and the first term of (6.38) is totally negligible with respect to the other terms, even when Γ is hardly larger than unity; for $\Gamma = 1$ we have the isothermal case, see section 6.2)

First order

$$K\Gamma(k_g^2 - \Lambda)\rho_{00}^{\Gamma-1}\eta_1 = 4\pi G\eta_1\rho_{00}$$

$a_1 \neq 0$ requires

$$k_g^2 = \Lambda + \frac{4\pi G}{K\Gamma\rho_{00}^{\Gamma-2}}$$

or to a very good approximation

$$k_g^2 = \frac{\Lambda\varphi_{00}}{K\Gamma\rho_{00}^{\Gamma-1}} \approx \frac{\Lambda\varphi_{00}}{v_s^2} \approx \frac{\Lambda\varphi_{00}m}{k_B T}. \tag{6.40}$$

In fact this corresponds to Jeans' criterion as $\Lambda\varphi_{00} \approx 4\pi G\rho_{00}$. At first sight this is surprising as we study here equilibria, not their stability. However, it is understandable that the equilibria use the same intrinsic unit of length as the stability.

Using reasonable values for the present day universe ($n = 10/\mathrm{m}^3$, $m = 1.6\ 10^{-27}$ kg, $T = 2.7$ K) we find $k_g^2 = 5\ 10^{-40}\mathrm{m}^{-2}$ and $\lambda_g = 3\ 10^{20}$ m or $3\ 10^4$ ly which is suitable for a large cluster or a galaxy, but not for the tessellation, which requires much larger lengths. However, one should not use the gas pressure, but rather the total pressure, including the radiation pressure, which is some 10^8 times larger, resulting in $\lambda_g \approx 10^8$ ly. Of course our analysis should then deal with the sum of the gas pressure and radiation pressure from the start. However, according to Chandrasekhar [58], the radiation behaves in this respect as if it has a polytropic exponent equal to 4/3. So the previous analysis works formally to a good approximation.

What happens if we go back in time? The Jeans' length, adapts itself, so that the Jeans mass remains the same.

Let us calculate more higher order terms in order to get an idea of the convergence of the series.

Second order

$$K\Gamma(4k_g^2 - \Lambda)\rho_{00}^{\Gamma-1}\left(\eta_2 + \frac{(\Gamma-2)}{2}\eta_1^2\right) = 4\pi G\eta_2\rho_{00}.$$

Using the previous approximations

$$\eta_2 = \frac{-2(\Gamma-2)}{3}\eta_1^2.$$

Third order

$$K\Gamma(9k_g^2 - \Lambda)\rho_{00}^{\Gamma-1}\left(\eta_3 + (\Gamma-2)\eta_1\eta_2 + \frac{(\Gamma-2)(\Gamma-3)}{6}\eta_1^3\right) = 4\pi G\eta_3\rho_{00}$$

or

$$\eta_3 = \frac{-9\left(\frac{-2(\Gamma-2)^2}{3} + \frac{(\Gamma-2)(\Gamma-3)}{6}\right)\eta_1^3}{8} = \frac{3(\Gamma-2)(3\Gamma-5)}{16}\eta_1^3$$

Fourth order

$$K\Gamma(16k_g^2 - \Lambda)\rho_{00}^{\Gamma-1}\left[\eta_4 + \frac{(\Gamma-2)}{2}\left(\eta_2^2 + 2\eta_1\eta_3\right) + \frac{(\Gamma-2)(\Gamma-3)}{2}\eta_1^2\eta_2 + \frac{(\Gamma-2)(\Gamma-3)(\Gamma-4)}{24}\eta_1^4\right] = 4\pi G\eta_4\rho_{00}$$

or

$$\eta_4 = \frac{16}{15}\frac{(\Gamma-2)}{2}\left[\frac{4}{9}(\Gamma-2)^2 + \frac{3}{8}(\Gamma-2)(3\Gamma-5) - \frac{2(\Gamma-2)(\Gamma-3)}{3} + \frac{(\Gamma-3)(\Gamma-4)}{12}\right]\eta_1^4$$

$$= \frac{(\Gamma-2)}{135}\left(17\Gamma^2 - 163\Gamma + 182\right)\eta_1^4$$

Comments

1. Here we may expect convergence for η_1 somewhat smaller than unity. (The terms seem to have alternating signs).
2. $\Gamma = 2$: Higher orders disappear and we remain with only one term, see section 6.3.1

6.4 Conclusion

Usually one considers a uniform equilibrium as a model of the universe. In this chapter we obtained a *non-uniform* solution: a specific cosinusoidal equilibrium for the polytropic exponent $\Gamma = 2$. Moreover this has another remarkable feature: its amplitude is arbitrary. This suggests that along the equilibria with varying amplitudes the stability is neutral or indifferent. A linearized perturbation analysis confirmed this view.

This non-uniform equilibrium may be a first step towards the so-called tessellation of the universe: observations indicate that the galaxies accumulate at certain (irregular) polyhedric walls and desert the interior of those polyhedra (cf. Jeans' criterion). Our numerical values rather indicate dimensions of the order of magnitude of galaxies, which is a good result in itself, but rather too small for the tessellation. However, using the suitable interpretation of the total pressure (gas pressure and radiation pressure) one may find dimensions of the order of the tessellation as well as of galaxies or clusters of galaxies.

Chapter 7

Nonuniform dynamic universe

7.1 Introduction

Instead of studying the stability of a unverse in equilibrium as in the previous chapters, we now consider an infinite, nonuniform gravitating medium in motion. Hence $\boldsymbol{v}_0 \neq 0$ in this chapter. However, $\boldsymbol{v}$ is still undetermined: which motions (steady state, oscillations or just evolution) are possible? We stick to our analysis with the one combined variable. It turns out that the cosmological constant is still necessary to avoid an empty universe in this analysis. The basic system of equations is as in section 5.2.2. We analyze such system (equations (5.1) - (5.4)) in the next section, however with $\boldsymbol{v}_0 \neq 0$.

7.2 Analysis of motions and perturbations

With the supposition that all quantities are functions of χ alone, then the set in (5.1) - (5.4) yields:

$$\omega\rho' + \boldsymbol{k}\cdot(\rho\boldsymbol{v})' = 0 \;\;\Rightarrow\;\; (\omega + \boldsymbol{k}\cdot\boldsymbol{v})\rho = \in, \tag{7.1}$$

$$\rho\,(\omega + \boldsymbol{v}\cdot\boldsymbol{k})\,\boldsymbol{v}' = -\boldsymbol{k}\,(p' + \rho\varphi')\,, \tag{7.2}$$

$$k^2\varphi'' + \Lambda\varphi = 4\pi G\rho, \tag{7.3}$$

$$p = K\rho^{\Gamma}. \tag{7.4}$$

After putting our basic system of equations in the form indicated above (equations (7.1) - (7.4)), we then investigate the cases when the constant, $\in$, is zero and when it is not zero.

7.2.1 Suppose that $\in = 0$

From equation (7.1), we have $\omega + \boldsymbol{k} \cdot \boldsymbol{v} = 0$ since ρ should be different from zero. Thus, splitting $\boldsymbol{v}$ in a component parallel to $\boldsymbol{k}$ and one perpendicular to $\boldsymbol{k}$, we have for the parallel component:

$$v_{\|} = \frac{\omega}{k}. \tag{7.5}$$

This represents just a uniform translation. This is irrelevant in our Newtonianlike approach (Galilean invariance) and may be left out. Note, however, that putting $v_{\|} = 0$ by applying a translation, might prevent $\in$ of remaining zero. However, equation (7.1) reads then $\omega\,\rho = \in$ and as ρ is now supposed nonuniform this requires $\omega = 0$ and $\in = 0$. We are then reduced to the equilibrium study of the previous chapter. Note that the perpendicular component, $v_{\perp}$, is arbitrary. However, if the density varies in the $\boldsymbol{k}$ direction only then arbitrary motions perpendicular to $\boldsymbol{k}$ do not affect the infinite state.

7.2.2 Suppose that $\in \neq 0$

The difference with the previous chapter is that there the left hand side of (7.2) was taken as zero for the equilibrium, as is usually done ($\boldsymbol{v}_0 = 0 = \boldsymbol{v}_0'$). This amounts to take $\in \boldsymbol{v}_0' = 0$. However, if we want to allow a dynamic universe we have to consider $\in \boldsymbol{v}_0' \neq 0$ which gives much wider possibilities than the restricted situation of the previous chapter, which allowed a single wavenumber k_0 only.

Consider a Fourier series for ρ:

$$\rho = \rho_0 \sum_{j=0}^{N} \eta_j e^{ij\chi} \tag{7.6}$$

with N going to infinity and constant coefficients η_j ($j = 0, 1, 2, \cdots$, and $\eta_0 = 1$) which have to be fixed by inserting (7.6) in the equations. We recall that χ contains the time, which constitutes the basic difference with the previous chapter, although formally some results look alike. As ρ may never become negative we have the condition, with $\rho_0 > 0$,

$$\sum_{j=0}^{N} \eta_j \geq 0. \tag{7.7}$$

Using a similar Fourier series for φ:

$$\varphi = \rho_0 \sum_{j=1}^{N} \varphi_j e^{ij\chi}, \tag{7.8}$$

with N going to infinity and φ_j constants to be fixed. We have left out the solution of the homogeneous part of (7.3) as this will induce, by (7.2), similar terms in v which, however, will find no match in equation (7.1). Upon substitution of (7.6) and (7.8) in (7.3) and identification one obtains

$$\left(\Lambda - j^2k^2\right)\varphi_j = 4\pi G\rho_0\eta_j, \qquad (j = 0, 1, 2, \cdots). \tag{7.9}$$

Note that for $j = 0$, we have $\Lambda\varphi_0 = 4\pi G\rho_0$, which requires $\Lambda \neq 0$.

Particular case: $\Gamma = 2$

We now restrict ourselves to the case $\Gamma = 2$.
Then, using (7.1) and (7.4), equation (7.2) becomes:

$$\in v' = -k\rho\,(2K\rho' + \varphi')\,. \tag{7.10}$$

Assuming again a Fourier series for $v_{||}$:

$$v_{||} = \sum_{j=0}^{N} v_j e^{ijx}, \tag{7.11}$$

while $v_\perp$ remains arbitrary and in fact irrelevant. Thus we obtain from equation (7.10)

$$\in \sum_{j=1}^{N} j v_j e^{ijx} = -\rho_0^2 k \left(\sum_{j=0}^{N} \eta_j e^{ijx}\right)\left[\sum_{j=1}^{N} j\,\Psi_j\,\eta_j\,e^{ijx}\right], \tag{7.12}$$

where

$$\Psi_j = \left[\left(2K + \frac{4\pi G}{\Lambda - j^2k^2}\right)\right], \qquad j = 1,\ 2,\ 3, \cdots.$$

It is at once clear that v_0 is still undetermined, but as this amounts to a uniform translation we may omit it. Next we have for the first order:

$$\in v_1 = -k\rho_0^2\eta_1\,\Psi_1 \tag{7.13}$$

and for the second to fifth order:

$$2 \in v_2 = -\rho_0^2 k\left[2\eta_2\,\Psi_2 + \eta_1^2\,\Psi_1\right]. \tag{7.14}$$

$$3 \in v_3 = -\rho_0^2 k\left\{3\eta_3\,\Psi_3 + 2\eta_1\eta_2\,\Psi_2 + \eta_2\eta_1\,\Psi_1\right\}, \tag{7.15}$$

$$4 \in v_4 = -\rho_0^2 k\left\{4\eta_4\,\Psi_4 + 3\eta_1\eta_3\,\Psi_3 + 2\eta_2\eta_2\,\Psi_2 + \eta_3\eta_1\,\Psi_1\right\} \tag{7.16}$$

and

$$5 \in v_5 = -\rho_0^2 k(5\Psi_5\eta_5 + 4\Psi_4\eta_4\eta_1 + 3\Psi_3\eta_3\eta_2 + 2\Psi_2\eta_2\eta_3 + \Psi_1\eta_1\eta_4). \tag{7.17}$$

Rewriting (7.1) as

$$kv_{\parallel} = \frac{\in}{\rho} - \omega, \tag{7.18}$$

yields for $v_0 = 0$:

$$\in = \omega\rho_0 \tag{7.19}$$

and thus

$$kv_{\parallel} = \omega\left[\left(\sum_{j=0}^{N}\eta_j e^{ijx}\right)^{-1} - 1\right]. \tag{7.20}$$

For the first order, we have from (7.20)

$$kv_1 = -\omega\eta_1. \tag{7.21}$$

Combining (7.21) with (7.13), one obtains

$$\frac{k^2\rho_0^2\eta_1}{\in}\,\Psi_1 = \omega\eta_1$$

or, requiring that $\eta_1 \neq 0$,

$$\omega^2 = k^2\rho_0\,\Psi_1 \tag{7.22}$$

which corresponds to the dispersion relation of Jeans, generalized to the case with cosmological constant. However, now it is a relation between ω (or σ) and k for a dynamical evolution instead of one indicating the stability or instability. The fact that both dispersion relations coincide is probably due to the fact that it was possible to choose $v_0 = 0$, without loss of generality, so that no new zero order quantity appeared. Thus the difference between a perturbation analysis and a study of the evolution becomes artificial. As $\Gamma = 2$ and neglecting Λ we have

$$\omega^2 = k^2\rho_0\left(2K - \frac{4\pi G}{k^2}\right) = k^2v_s^2 - 4\pi G\rho_0. \tag{7.23}$$

Next, for $j = 2$, we obtain from equation (7.20):

$$kv_2 = \omega\left(-\eta_2 + \eta_1^2\right). \tag{7.24}$$

Substituting equation (7.14) into equation (7.24) yields

$$-\frac{\rho_0^2k^2}{2\in}\left[2\eta_2\,\Psi_2 + \eta_1^2\,\Psi_1\right] = \omega\left(\eta_1^2 - \eta_2\right). \tag{7.25}$$

Using equation (7.22) in equation (7.25) we have

$$-\rho_0k^2\eta_2\,\Psi_2 - \frac{\eta_1^2\omega^2}{2} = \omega^2\left(\eta_1^2 - \eta_2\right)$$

or

$$\eta_2 = \frac{3}{2}\eta_1^2 \left[1 - \frac{k^2}{\omega^2}\left(v_s^2 + \frac{4\pi G\rho_0}{\Lambda - 4k^2}\right)\right]^{-1}. \tag{7.26}$$

Neglecting Λ yields

$$\eta_2 = \frac{-\eta_1^2\,\omega^2}{2\,\pi\,G\,\rho_0}. \tag{7.27}$$

For the third, fourth and fifth orders we respectively have from equation (7.20):

$$kv_3 = \omega\left(-\eta_3 + 2\eta_1\eta_2 - \eta_1^3\right), \tag{7.28}$$

$$kv_4 = \omega\left(-\eta_4 + 2\eta_1\eta_3 + \eta_2^2 - 3\eta_1^2\eta_2 + \eta_1^4\right) \tag{7.29}$$

and

$$k\,v_5 = \omega\left\{-\eta_5 + 2\eta_1\eta_4 + 2\eta_2\eta_3 - 3\eta_1\left(\eta_1\eta_3 + \eta_2^2\right) + 4\eta_1^3\eta_2 - \eta_1^5\right\} \tag{7.30}$$

Substituting for v_3, v_4 and v_5 we obtain respectively:

$$\eta_3 = \left[1 - \frac{\rho_0 k^2}{\omega^2}\Psi_3\right]^{-1}\left\{\frac{\rho_0 k^2}{3\omega^2}\left[\eta_1\eta_2\left(2\Psi_2 + \Psi_1\right)\right] + 2\eta_1\eta_2 - \eta_1^3\right\}, \tag{7.31}$$

$$\eta_4 = \left[1 - \frac{\rho_0 k^2}{\omega^2}\Psi_4\right]^{-1}\left\{\frac{\rho_0 k^2}{4\omega^2}\left[3\eta_1\eta_3\,\Psi_3 + 2\eta_2\eta_2\,\Psi_2 + \eta_3\eta_1\,\Psi_1\right]\right.$$
$$\left. + \left(2\eta_1\eta_3 + \eta_2^2 - 3\eta_1^2\eta_2 + \eta_1^4\right)\right\}. \tag{7.32}$$

and

$$\eta_5 = \left[1 - \frac{\rho_0 k^2}{\omega^2}\Psi_5\right]^{-1}\left\{\eta_1\left[2\eta_4 - 3\left(\eta_1\eta_3 + \eta_2^2\right) + 4\eta_1^2\eta_2 - \eta_1^4\right]\right.$$
$$\left. + \frac{\rho_0 k^2}{5\omega^2}\left(4\Psi_4\eta_4\eta_1 + 3\Psi_3\eta_3\eta_2 + 2\Psi_2\eta_2\eta_3 + \Psi_1\eta_1\eta_4\right)\right\}. \tag{7.33}$$

Neglecting Λ and substituting appropriate equations yield

$$\eta_3 = \frac{9\omega^2\eta_1^3}{32\rho_0\left(\pi G\right)^2}\left(3k^2K - 4\pi G\right) \tag{7.34}$$

$$\eta_4 = -\frac{\omega^2 \eta_1^4}{12\rho_0 (\pi G)^3}\left(21k^4K^2 - 54\pi GKk^2 + 32\pi^2G^2\right), \tag{7.35}$$

Similarly, one may obtain an expression for η_5 even those for higher orders, but calculations are more involving. From the determined expressions we notice that, when $\omega \approx 0$ (near Jeans' critical wavelength) we have $\eta_2 = \eta_3 = \eta_4 = \cdots \approx 0$ implying very good convergence. For $\omega = 0$ we have no higher orders terms and for $\eta_1 = 0$, trivial solution is obtained.

7.3 Conclusion

Although formally the present analysis of the evolution of a dynamic inhomogeneous turns out to be formally quite similar to a previous analysis, the interpretation is different. More over an infinity of wavenumbers is now possible while in the previous chapter, with a strict interpretation of the equilibrium ($\boldsymbol{v}_0 = \mathbf{0}$ and $\boldsymbol{v}_0' = \mathbf{0}$), only one k_0 was allowed.

If one starts from a static universe and perturbs it, the perturbation amplitude (say A) has to be very small: relative to the universe, the explosion of a supernova, whatever gigantic this look to us, is to be considered as "infinitely small". However, in view of the big bang one should allow a really dynamic universe with huge velocities from the start. The situation is different for e.g. a laboratory plasma: there one may start with a quite equilibrium and suddenly apply a strong external perturbation. Cf our suggestion to do such experiments e.g. in a Q-machine.

Chapter 8

Nonlinear stability analysis of an infinite homogeneous gravitating medium with a cosmological term pervaded by a homogeneous magnetic field

8.1 Introduction

We have studied previously an infinite homogeneous medium using the Newtonian gravitation supplemented with a cosmological term in order to start with the correct equilibrium. There two distinct applications possible: One is for the universe, and then the cosmological constant may play a real role. It may be recalled that the interest in the cosmological constant is revived strongly in view of the recent observations with supernovae used as a kind of standard candles to look back far in the history of the universe.

The other application concerns large gravitating media like interstellar clouds and galaxies. In that case, as the cosmological constant is extremely small, it will play no role except in making the equilibrium consistent. The cosmological constant may then be approximated by zero in all perturbation terms. In view of the consistency and of both applications we elaborate the theory with cosmological term.

Next we assume that the (ionized) matter, supposed to be perfectly conducting, is pervaded by a homogeneous magnetic field. In the case of the universe one may wonder whether a magnetic field is actually present. However, as magnetic fields do occur at all other scales of gravitating bodies, the hypothesis seems plausible. Moreover, the Fermi acceleration in our galaxy can not provide or contain particles with energies above say 10^{15}eV. As there are still particles (although less numerous) following the distribution based on the Fermi acceleration mechanism up to the highest energies measured (10^{22}eV), the hypothesis of a very weak magnetic field is highly probable. Its induction is probably smaller than 10^{-10} tesla (the value in our galaxy is around 2 10^{-10} tesla); yet in view of the huge spaces involved this may create the cosmic radiation of the most extreme energies.

8.2 Basic equations

Obvious the basic system of equations consists of the Poisson equation for gravitation, supplemented by a cosmological term, the equation of hydrodynamics and the equations of Maxwell. Clearly some simplifications are required to handle such an impressive system.

We assume that the material equations are $\boldsymbol{B} = \mu \boldsymbol{H}$ and $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ (cf. equations (4.7) and (4.8)) with μ and ε being constant. In a homogeneous medium this is quite plausible. Moreover the variations in μ are always extremely small except in ferromagnetism, which is not under consideration here, so that we may completely neglect variations of μ in the perturbed situation too. The constancy of ε in the presence of a magnetic field and perturbations (which we allow to be quite large) may be somewhat less adequate, but as we will use the magnetohydrodynamic (MHD) approach in which $\boldsymbol{E}$ and $\boldsymbol{D}$ do not occur anymore explicitly in the reduced system of equations, the value of ε is irrelevant, except if one wants to calculate $\boldsymbol{E}$ and $\boldsymbol{D}$ a posterior; i.e. after having solved the reduced system.

The resistivity and the viscosity are taken to be zero. The MHD approximation of the electric current is

$$\boldsymbol{j} = \operatorname{rot} \boldsymbol{H}. \tag{8.1}$$

Hence our basic (reduced) system of equations consists of the equation of continuity, motion, Poisson (including Λ), polytropics, conservation of magnetic flux and evolution of the magnetic field:

$$\partial_t \rho + \operatorname{div} \rho \boldsymbol{v} = 0, \tag{8.2}$$

$$\rho \frac{\mathrm{d} \boldsymbol{v}}{\mathrm{d} t} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \varphi + \mu (\operatorname{rot} \boldsymbol{H}) \times \boldsymbol{H}, \tag{8.3}$$

$$\Delta \varphi + \Lambda \varphi = 4 \pi G \rho, \tag{8.4}$$

$$p = K \rho^{\Gamma}, \tag{8.5}$$

$$\nabla \cdot \boldsymbol{H} = 0, \tag{8.6}$$

$$\partial_t \boldsymbol{H} = \mathrm{rot}(\boldsymbol{v} \times \boldsymbol{H}), \tag{8.7}$$

where ρ is the mass density (the electric charges do not occur explicitly in the MHD approximation), $\boldsymbol{v}$ is the velocity, p the pressure, φ the gravitational potential, Λ the cosmological constant, G the gravitational constant, K is a constant and Γ is the polytropic exponent (supposed constant) which represents the variations in the state of the matter (isothermal: $\Gamma = 1$, adiabatic: $\Gamma = \gamma$, incompressible: $\Gamma = \infty$).

Using vector identities and (8.6) one may rewrite equations (8.3) and (8.7) as

$$\rho\,(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) = -\nabla\left(p + \frac{\mu H^2}{2}\right) - \rho \nabla \varphi + \mu\, \boldsymbol{H} \cdot \nabla \boldsymbol{H} \tag{8.8}$$

$$\partial_t \boldsymbol{H} = (\boldsymbol{H} \cdot \nabla)\, \boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{H} - (\mathrm{div}\, \boldsymbol{v})\, \boldsymbol{H}. \tag{8.9}$$

For the equilibrium we put $\boldsymbol{v} = 0$, $\rho = \rho_0 =$ constant and $\boldsymbol{H} = \boldsymbol{H}_0$, a constant homogeneous field. Then $p_0 = K\,\rho_0^\Gamma$ and

$$\varphi_0 = \frac{4\,\pi\, G\, \rho_0}{\Lambda} \tag{8.10}$$

are constants too, independent of space ($\boldsymbol{r}$) and time (t). Obviously equation (8.10) requires $\Lambda \neq 0$.

8.3 Stability analysis

8.3.1 Combined variable and preliminary elimination

Introducing the combined variable $\chi = \omega\, t + \boldsymbol{k} \cdot \boldsymbol{r}$ so that $\partial_t = \omega$ and $\nabla = \boldsymbol{k}$. An accent denotes derivation with respect to χ. The system reduces to equation (8.5) and

$$\omega \rho' + \boldsymbol{k} \cdot (\rho\, \boldsymbol{v})' = 0, \tag{8.11}$$

$$\rho\,(\omega + \boldsymbol{k} \cdot \boldsymbol{v})\, \boldsymbol{v}' = -\,(p' + \mu\, \boldsymbol{H} \cdot \boldsymbol{H}' + \rho \varphi')\, \boldsymbol{k} + \mu\,(\boldsymbol{k} \cdot \boldsymbol{H})\, \boldsymbol{H}', \tag{8.12}$$

$$k^2 \varphi'' + \Lambda\, \varphi = 4\, \pi\, G\, \rho, \tag{8.13}$$

$$\boldsymbol{k} \cdot \boldsymbol{H}' = 0, \tag{8.14}$$

$$\omega \boldsymbol{H}' = (\boldsymbol{k} \cdot \boldsymbol{H})\, \boldsymbol{v}' - (\boldsymbol{k} \cdot \boldsymbol{v})\, \boldsymbol{H}' - (\boldsymbol{k} \cdot \boldsymbol{v}')\, \boldsymbol{H}. \tag{8.15}$$

Integrating equation (8.11) yields

$$(\omega + \boldsymbol{k} \cdot \boldsymbol{v})\, \rho = \epsilon = \omega\, \rho_0 \tag{8.16}$$

as $\rho = \rho_0$ for $\boldsymbol{v} = \boldsymbol{v}_0 = 0$. Integrating equation (8.14) yields

$$\boldsymbol{k} \cdot \boldsymbol{H} = \in = \boldsymbol{k} \cdot \boldsymbol{H}_0. \tag{8.17}$$

Using (8.16) and (8.17) allows to simplify equations (8.15) and (8.12):

$$\frac{\omega\,\rho_0}{\rho} \boldsymbol{H}' = (\boldsymbol{k} \cdot \boldsymbol{H}_0)\, \boldsymbol{v}' - (\boldsymbol{k} \cdot \boldsymbol{v}')\, \boldsymbol{H}. \tag{8.18}$$

$$\omega\, \rho_0\, \boldsymbol{v}' = -\,(p' + \mu\, \boldsymbol{H} \cdot \boldsymbol{H}' + \rho\varphi')\, \boldsymbol{k} + \mu\, (\boldsymbol{k} \cdot \boldsymbol{H}_0)\, \boldsymbol{H}'. \tag{8.19}$$

8.3.2 The incompressible case

If the media were incompressible, $\operatorname{div} \boldsymbol{v} = 0$; then, multiplying equation (8.19) scalarly by $\boldsymbol{k}$, yields:

$$0 = -\,(p' + \mu\, \boldsymbol{H} \cdot \boldsymbol{H}' + \rho\varphi')\, k^2 \tag{8.20}$$

and as the case $k = 0$ is rather trivial, (8.18) and (8.19) reduce to:

$$\omega\, \boldsymbol{H}' = (\boldsymbol{k} \cdot \boldsymbol{H}_0)\, \boldsymbol{v}', \tag{8.21}$$

$$\omega\, \rho_0\, \boldsymbol{v}' = \mu\, (\boldsymbol{k} \cdot \boldsymbol{H}_0)\, \boldsymbol{H}', \tag{8.22}$$

where we have used $\rho = \rho_0$ as we are now dealing with the homogeneous, incompressible case. The compatibility condition of equations (8.21) and (8.22) leads immediately to

$$\omega^2 = \mu\, \frac{(\boldsymbol{k} \cdot \boldsymbol{H}_0)^2}{\rho_0} = \omega_A^2, \tag{8.23}$$

which is the well-known dispersion relation of Alfvén; unchanged although we have included gravitation: cf. comment below.

Comments

1. General solution: linear problem!
   Except for the approximations in the basic equations, all we used up to now is that the functions depend only on the combined variable in χ, nothing more: no Fourier analysis, no perturbation. The equations (8.21) and (8.22) or rather (8.22) and (8.23) contain the general answer. This means once $\boldsymbol{k}$, ρ_0 and $\boldsymbol{H}_0$ are chosen (i.e., k, ρ_0, H_0 and the angle θ between $\boldsymbol{k}$ and $\boldsymbol{H}_0$) that ω is fixed by (8.23) (up to an irrelevant sign) and that $\boldsymbol{v}$ and $\boldsymbol{H}$ are related by (8.21) or (8.22) or

$$\sqrt{\rho_0}\, \boldsymbol{v}' = \pm\sqrt{\mu}\, \boldsymbol{H}' \tag{8.24}$$

or upon integration

$$\sqrt{\rho_0}\, \boldsymbol{v} = \pm\sqrt{\mu}\, (\boldsymbol{H} - \boldsymbol{H}_0)\,. \tag{8.25}$$

Any amplitude for e.g. $\boldsymbol{v}$ is allowed; $\boldsymbol{H}$ takes the amplitude corresponding to (8.25). In fact the problem has been reduced to a linear one.

2. Equipartition of the perturbed energies

   (a) Squaring equation (8.25) yields

   $$\rho_0\, v^2 = \mu\, (\boldsymbol{H} - \boldsymbol{H}_0)^2 \tag{8.26}$$

   and thus the kinetic energy of the perturbation is equal to the magnetic energy density of the perturbation, a nice example of equipartition energy. Chandrasekhar [9] mentioned this, but that was only in first order of the perturbation. Here it is for an arbitrary one. Notice that the wave and the field both correspond to one degree of freedom.

   (b) As ω/k is the phase velocity v_{ph} we may interpret equation (8.23) as follows:

   $$\rho_0\, v_{ph}^2 = \mu\, H_0^2. \tag{8.27}$$

   Thus energy density which one may virtually associate with the wave phenomena corresponding to the magnetic energy density of the equilibrium. A nice interpretation of the dispersion relation and a second nice example of equipartition of energy!

3. Gravitation is kept out of the analysis

   Formally gravitation was included from the outset. However in equations (8.21) or (8.22) and (8.23), and in equation (8.16) and (8.17) too, gravitation is not involved. The problem is solved by a subset and gravitation is not involved in it. Looking for the effect on gravitation once the subset is solved leads nowhere. In fact we have no grip on the gravitational part: in equation (8.20) p' approaches infinity as Γ does so for incompressibility. Hence equation (8.20) is useless for the gravitation, which leaves us with (8.13) as the only equation relevant to gravitation, completely general and not specified in any way to instability. However, there is nothing inconsistent in the whole situation: for incompressible matter we do not expect instabilities or oscillations; in fact they have vanishing amplitude. There are no sound waves or gravito-acoustic waves, only MHD waves, more precisely Alfvén waves.

8.3.3 Compressible case

However, the interstellar or intergalactic medium can hardly be considered as incompressible and this complicates the analysis strongly.

From (8.11) or (8.16) we have

$$\boldsymbol{k}\cdot\boldsymbol{v}' = -\frac{\omega\,\rho_0\,\rho'}{\rho^2}, \tag{8.28}$$

and equation (8.18) may then be rewritten as

$$\omega\,\rho_0\left(\frac{\boldsymbol{H}'}{\rho}-\frac{\rho'\,\boldsymbol{H}}{\rho^2}\right) = (\boldsymbol{k}\cdot\boldsymbol{H}_0)\,\boldsymbol{v}'. \tag{8.29}$$

Integration yields

$$\omega\,\rho_0\left(\frac{\boldsymbol{H}}{\rho}-\frac{\boldsymbol{H}_0}{\rho_0}\right) = (\boldsymbol{k}\cdot\boldsymbol{H}_0)\,\boldsymbol{v} \tag{8.30}$$

as $\boldsymbol{v}_0 = 0$. Equation (8.30) may be interpreted as an alternative form of the frozen-in theorem: matter and field move together.

Multiplying (8.19) scalarly by $\boldsymbol{k}$ and using (8.28) yields

$$\frac{\omega^2\,\rho_0^2\,\rho'}{\rho^2} = (p' + \mu\,\boldsymbol{H}\cdot\boldsymbol{H}' + \rho\varphi')\,k^2 \tag{8.31}$$

which generalizes (8.20) to take compressibility into account. Equation (8.31) allows us to simplify equation (8.19):

$$\omega\,\rho_0\,\boldsymbol{v}' = -\frac{\omega^2\,\rho_0^2\,\rho'}{\rho^2\,k^2}\boldsymbol{k} + \mu\,(\boldsymbol{k}\cdot\boldsymbol{H}_0)\,\boldsymbol{H}'. \tag{8.32}$$

Integration yields

$$\omega\,\rho_0\,\boldsymbol{v} = \frac{\omega^2\,\rho_0}{\rho\,k^2}\,(\rho_0-\rho)\,\boldsymbol{k} + \mu\,(\boldsymbol{k}\cdot\boldsymbol{H}_0)\,(\boldsymbol{H}-\boldsymbol{H}_0)\,. \tag{8.33}$$

$\boldsymbol{H}-\boldsymbol{H}_0$ is perpendicular to $\boldsymbol{k}$ according to equation (8.17). Thus $\boldsymbol{v}$, like $\boldsymbol{v}'$ (as was already clear from (8.19)), has a component parallel to $\boldsymbol{k}$ and one perpendicular to $\boldsymbol{k}$.

$$v_{\parallel} = \frac{\omega\,(\rho_0-\rho)}{\rho\,k}, \qquad v_{\perp} = \frac{\mu\,(\boldsymbol{k}\cdot\boldsymbol{H}_0)}{\omega\,\rho_0}\,(\boldsymbol{H}-\boldsymbol{H}_0)\,. \tag{8.34}$$

Elimination of $\boldsymbol{H}$ between (8.30) and (8.33) yields

$$\frac{1}{\omega}\left(\omega^2\,\rho_0 - \omega_A^2\,\rho\right)\,\boldsymbol{v} = (\rho_0-\rho)\left[\frac{\omega^2\,\rho_0}{\rho\,k^2}\boldsymbol{k} - \frac{\mu\,(\boldsymbol{k}\cdot\boldsymbol{H}_0)}{\rho_0}\boldsymbol{H}_0\right]. \tag{8.35}$$

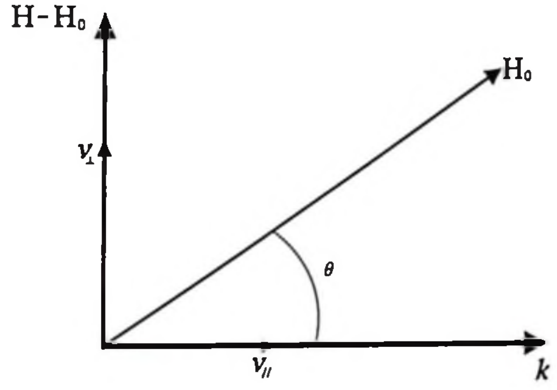

Figure 8.1: *The figure showing a graph of* $\boldsymbol{H}_0$ *on the* $\boldsymbol{H}-\boldsymbol{H}_0$ *vs* $\boldsymbol{k}$ *plane. Note that* $\boldsymbol{v}_{\|}$ *points in the same direction as* $\mathbf{k}$ *when* $\rho < \rho_0$ *and in the opposite direction when* $\rho > \rho_0$.

This expresses $\boldsymbol{v}$ in terms of ρ alone (and the constant quantities). In particular it is clear that $\boldsymbol{v}$ is coplanar with $\boldsymbol{k}$ and $\boldsymbol{H}_0$ and so is $\boldsymbol{H}$. It may be noted that upon scalar multiplication of (8.35) by $\boldsymbol{k}$ and eliminating $\boldsymbol{k}\cdot\boldsymbol{v}$ then in (8.16) leads to an identity.

With θ the angle between $\boldsymbol{k}$ and $\boldsymbol{H}_0$ (see figure 8.1) and an obvious notation:

$$\boldsymbol{H}_0 = H_0 \cos\theta \boldsymbol{1}_k + H_0 \sin\theta \boldsymbol{1}_\perp \tag{8.36}$$

we obtain from (8.33) again the first equation (8.34) and

$$\boldsymbol{v}_\perp = \frac{-\mu\,\omega\,(\rho_0-\rho)\,(\boldsymbol{k}\cdot\boldsymbol{H}_0)}{\rho_0\,(\omega^2\,\rho_0-\omega_A^2\,\rho)} H_0 \sin\theta \boldsymbol{1}_\perp, \tag{8.37}$$

which is an alternative expression for the second equation (8.34). Elimination of $\boldsymbol{v}_\perp$ leads to

$$\boldsymbol{H}-\boldsymbol{H}_0 = -\frac{\omega^2\,(\rho_0-\rho)\,H_0\sin\theta\boldsymbol{1}_\perp}{(\omega^2\,\rho_0-\omega_A^2\,\rho)}, \tag{8.38}$$

which is an alternative expression for $\boldsymbol{H}$. Indeed eliminating $\boldsymbol{v}$ between (8.30) and (8.35) yields

$$\left(\omega^2\,\rho_0-\omega_A^2\,\rho\right)\boldsymbol{H} = \omega^2\,(\boldsymbol{k}\cdot\boldsymbol{H}_0)\,(\rho_0-\rho)\,\frac{\boldsymbol{k}}{k^2} + \rho\left(\omega^2-\omega_A^2\right)\boldsymbol{H}_0. \tag{8.39}$$

Note that in the incompressible case $\rho=\rho_0$ yields at once the Alfvén dispersion relation (8.23).

Comments

1. On the consistency of the MHD approximation
   It is not evident a priori that $\boldsymbol{v}$, $\boldsymbol{k}$, $\boldsymbol{H}_0$ and $\boldsymbol{H}$ are coplanar, although may be expected: As density waves propagate along the wave vector in the absence of magnetic field and on the other hand motions are unimpended along $\boldsymbol{H}$. Thus a combination of the two specific directions is not improbable. A deep analysis reveals some inconsistency in the treatment, as had to be expected in the MHD approximation. Indeed, equation (8.1) or
$$\boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{H}' \tag{8.40}$$
   shows that $\boldsymbol{j}$ is perpendicular to $\boldsymbol{k}$. Now the current is parallel to $\boldsymbol{v}_e$, the velocity due to the charges. We may expect $\boldsymbol{v}_e$ to be parallel to the mass velocity $\boldsymbol{v}$ (as the positive and negative charges move along the same axis, although mostly in opposite directions). Thus we infer $\boldsymbol{k} \cdot \boldsymbol{v} = 0$ at least as an approximation. Clearly the contribution of the compressibility has to be linked to the omitted term $\varepsilon \partial_t \boldsymbol{E}$ in equation (8.1) and thus to $\varepsilon \omega \boldsymbol{E}'$ which is neglected in equation (8.40). Of course, this term should then have been taken into account right from the beginning, which, however, would have complicated the analysis a lot.

2. Compressibility leads to a nonlinear analysis
   It was stated in the comments in the incompressible case that the analysis was quite exceptionally reduced to a linear system and that any amplitude for $\boldsymbol{v}$ (or $\boldsymbol{H}$) was allowed. The compressibility prevents this. We have to linearize to obtain a dispersion relation. Using equations (8.5) and (8.38) to eliminate p' and $\boldsymbol{H} \cdot \boldsymbol{H}'$ in equation (8.31) yields
$$\begin{aligned}\frac{\omega^2 \rho_0^2 \rho'}{\rho^2} &= \left[p' + \rho\varphi' + \mu\left(\boldsymbol{H}_0 - \frac{\omega^2(\rho_0 - \rho)\,H_0 \sin\theta \boldsymbol{1}_\perp}{(\omega^2\rho_0 - \omega_A^2\rho)}\right)\right. \\ &\quad \left.\times\left(\frac{\omega^2(\omega^2-\omega_A^2)\,\rho_0 H_0 \sin\theta \boldsymbol{1}_\perp}{(\omega^2\rho_0 - \omega_A^2\rho)^2}\rho'\right)\right]k^2 \\ &= \left[K\Gamma\rho^{\Gamma-1}\rho' + \rho\varphi' + \frac{\omega^2(\omega^2-\omega_A^2)^2\rho_0\rho\mu H_0^2 \sin^2\theta\rho'}{(\omega^2\rho_0 - \omega_A^2\rho)^3}\right]k^2.\end{aligned} \tag{8.41}$$
   Using equation (8.39) instead of (8.38) confirms (8.41). Now we split the analysis in two: once without gravitation and once with gravitation.

8.3.3.1 Without gravitation: just compressible Alfvén waves?

When φ is left out the elimination is completed: equation (8.41) contains only ρ. The incompressible case was treated in section 8.3.2, hence we may divide by $\rho' \neq 0$ and we are left with an ordinary algebraic equation:

$$\left(\omega^2\rho_0^2 - K\Gamma\rho^{\Gamma+1}k^2\right)\left(\omega^2\rho_0 - \omega_A^2\rho\right)^3 = \omega^2\omega_A^2\left(\omega^2 - \omega_A^2\right)^2\rho_0^2\rho^3\tan^2\theta. \quad (8.42)$$

Omitting the term in Γ, we obtain a cubic equation which we are able to solve at once:

$$\frac{\omega^2\rho_0}{\omega_A^2\rho} = 1 + \left[\left(\frac{\omega^2}{\omega_A^2} - 1\right)^2\tan^2\theta\right]^{1/3}. \quad (8.43)$$

It is clear that ω_A may be used as the unit for ω and ρ_0 as the unit for ρ. Only the real root (positive) of the cubic root gives oscillations. It is clear that if $\omega^2 < \omega_A^2$ then $\rho < \rho_0$. If $\omega = \pm\omega_A$ then $\rho = \rho_0$.

However, the main conclusion from (8.43), or generally (8.42), is that ρ has to be constant. Suppose ρ should evolve from ρ_0 to another constant value: it may not do this as a function of χ alone or otherwise some of the approximations involved in the analysis are too crude and prevent a consistent picture. (Note that the conservation of mass does not prevent a (de)compression and thus an increase or decrease of ρ as the medium is infinite.) Evolution is still possible, but the variation in time and space may at most start as a function of χ. Precisely for this start we may replace ρ by ρ_0 (or by $\rho_0 + \delta\rho$) in equation (8.42). For the equilibrium equation we obtain from (8.42), after dividing by ρ_0^5:

$$\left(\omega^2 - v_s^2k^2\right)\left(\omega^2 - \omega_A^2\right)^3 = \omega^2\omega_A^2\left(\omega^2 - \omega_A^2\right)^2\tan^2\theta \quad (8.44)$$

with $v_s = (\Gamma p_0/\rho_0)^{1/2}$ the sound velocity. A double solution is obviously $\omega^2 = \omega_A^2$, corresponding to the situation that the matter behaves as if it were incompressible, although it is supposed to allow for compressibility. Next, we obtain a biquadratic equation:

$$\omega^4 - \omega^2\left(\frac{\omega_A^2}{\cos^2\theta} + v_s^2k^2\right) + v_s^2\omega_A^2k^2 = 0 \quad (8.45)$$

which is the dispersion relation for Alfvén waves in a compressible medium as obtained by Chandrasekhar [9], however by doing a linearized perturbation analysis on a system of equations while here we performed the elimination to obtain a single equation, valid for any motion, depending on χ only. Our analysis shows a similarity with the sound waves: the dispersion relation may be obtained from the fully integrated equation, but the evolution faces singularities in all higher order terms if one sticks to the use of the combined variable χ only.

Alternative procedure

In the absence of φ we may integrate equation (8.31):

$$\frac{\omega^2\rho_0\,(\rho-\rho_0)}{\rho}=\left\{p-p_0+\frac{\omega^2\big[2\omega_A^2\rho-\omega^2(\rho_0+\rho)\big](\rho_0-\rho)\mu H_0^2\sin^2\theta}{2\left(\omega^2\rho_0-\omega_A^2\rho\right)^2}\right\}k^2$$

or $$\frac{\omega^2\rho_0}{\rho}=Kk^2\left(\frac{\rho^\Gamma-\rho_0^\Gamma}{\rho-\rho_0}\right)+\frac{\omega^2\left[2\omega_A^2\rho-\omega^2(\rho_0+\rho)\right]\rho_0\omega_A^2\tan^2\theta}{2\left(\omega^2\rho_0-\omega_A^2\rho\right)^2}. \tag{8.46}$$

Omitting the pressure term now leads to a quadratic equation for ρ. Anyway, with or without the pressure term, ρ has to be constant and we recover the same conclusion as above. Let us try to recover the dispersion relation as well by putting $\rho=\rho_0+\delta\rho$, with $\delta\rho\to 0$

$$\omega^2=v_s^2k^2+\frac{\omega^2\omega_A^2\tan^2\theta}{\omega^2-\omega_A^2}, \tag{8.47}$$

which coincides with equation (8.45) and confirms again the dispersion relation derived by Chandrasekhar in spite of the objections we have made.

The contrast with the incompressible Alfvén waves is great: there was no inconsistency in the assumptions and any motion in χ, satisfying $\omega^2=\omega_A^2$, was allowed ($\boldsymbol{H}$ was determined once $\boldsymbol{v}$ was chosen). In the compressible case, which in general allows more motions, we face a (minor) inconsistency in the basic set and no motion at all may be described by using the combined variable χ, although the dispersion relation follows at once from the final equations, either (8.42) or (8.46). Another difference with the incompressible case will show up in the extension to gravity, which now makes sense - and changes the whole situation again (see below).

8.3.3.2 Including gravitation

In section 8.3.3.1, gravitation was omitted leading φ to be dropped out of equation (8.41). But now, since gravitation is included in our analysis, we keep the φ. Hence multiplying on both sides of equation (8.41) by $1/(\rho k^2)$ and rearranging, we get:

$$\varphi'=\frac{\omega^2\rho_0^2\rho'}{k^2\rho^3}-K\Gamma\rho^{\Gamma-2}\rho'-\frac{\omega^2\left(\omega^2-\omega_A^2\right)^2\rho_0^2\omega_A^2\tan^2\theta\rho'}{k^2\left(\omega^2\rho_0-\omega_A^2\rho\right)^3}, \tag{8.48}$$

which can be integrated to give

$$\varphi=\frac{4\pi G\rho_0}{\Lambda}+\frac{\omega^2}{2k^2}\left(1-\frac{\rho_0^2}{\rho^2}\right)+\frac{K\Gamma\left(\rho_0^{\Gamma-1}-\rho^{\Gamma-1}\right)}{\Gamma-1}$$

$$+\frac{\omega^2 \tan^2\theta}{2k^2}\left(1-\frac{\left(\omega^2-\omega_A^2\right)^2\rho_0^2}{\left(\omega^2\rho_0-\omega_A^2\rho\right)^2}\right). \tag{8.49}$$

Differentiating equation (8.48) with respect to χ we get

$$\varphi'' = \frac{\omega^2\rho_0^2}{k^2}\left(\frac{\rho''}{\rho^3}-3\frac{\rho'^2}{\rho^4}\right) - K\Gamma\rho^{\Gamma-2}\left[\rho''+(\Gamma-2)\,\rho^{-1}\rho'^2\right]$$
$$-\frac{\omega^2\left(\omega^2-\omega_A^2\right)^2\rho_0^2\omega_A^2\tan^2\theta}{k^2\left(\omega^2\rho_0-\omega_A^2\rho\right)^3}\left(\rho''+\frac{3\,\omega_A^2\rho'^2}{\left(\omega^2\rho_0-\omega_A^2\rho\right)}\right), \tag{8.50}$$

Then putting equations (8.49) and (8.50) into (8.13) yields

$$\omega^2\rho_0^2\left(\frac{\rho''}{\rho^3}-3\frac{\rho'^2}{\rho^4}\right) - k^2 K\,\Gamma\rho^{\Gamma-2}\left[\rho''+(\Gamma-2)\,\rho^{-1}\rho'^2\right]$$
$$-\frac{\omega^2\left(\omega^2-\omega_A^2\right)^2\rho_0^2\omega_A^2\tan^2\theta}{\left(\omega^2\rho_0-\omega_A^2\rho\right)^3}\left(\rho''+\frac{3\,\omega_A^2\rho'^2}{\left(\omega^2\rho_0-\omega_A^2\rho\right)}\right)$$
$$+4\pi G\rho_0+\frac{\omega^2\Lambda}{2k^2}\left(1-\frac{\rho_0^2}{\rho^2}\right)+\frac{K\,\Gamma\,\Lambda\left(\rho_0^{\Gamma-1}-\rho^{\Gamma-1}\right)}{\Gamma-1}$$
$$+\frac{\omega^2\Lambda\tan^2\theta}{2k^2}\left(1-\frac{\left(\omega^2-\omega_A^2\right)^2\rho_0^2}{\left(\omega^2\rho_0-\omega_A^2\rho\right)^2}\right)=4\pi G\,\rho, \tag{8.51}$$

which may be used in the determination of the higher orders coefficients. For example, linearizing equation (8.51) one obtains:

$$\left(\omega^2-\omega_A^2\right)\left\{\left(\omega^2-k^2v_s^2\right)\rho_1''+\frac{\left(\omega^2-k^2v_s^2\right)\Lambda\,\rho_1}{k^2}\right.$$
$$\left.-\left(\rho_1''+\frac{\Lambda\,\rho_1}{k^2}\right)\left(\frac{\omega^2\omega_A^2\tan^2\theta}{\omega^2-\omega_A^2}\right)-4\pi\,G\,\rho_0\rho_1\right\}=0. \tag{8.52}$$

Then assuming that ρ_1 varies as $e^{i\chi}$, we either get $\omega^2=\omega_A^2$ or

$$\omega^4-\omega^2\left(\frac{\omega_A^2}{\cos^2\theta}+k^2v_s^2+\frac{4\pi\,G\,\rho_0\,k^2}{\Lambda-k^2}\right)+k^2\omega_A^2\left(v_s^2+\frac{4\pi\,G\,\rho_0}{\Lambda-k^2}\right)=0. \tag{8.53}$$

This generalizes the dispersion relation of Chandrasekhar [9] for compressible Alfvén waves in an infinite homogeneous universe with Newtonian gravitation, made consistent by adding a cosmological term. If $G=0$, or gravitation is left out, we recover from (8.53), equation (8.45) or equation (8.47), i.e. the dispersion relation for the compressible Alfvén waves without gravitation.

Alternative procedures

Integrating equation (8.13) and substituting equations (8.48) and (8.49), one gets the first order differential equation:

$$\frac{\omega^2\,\rho_0^2\,\rho'}{\rho^3} - k^2 K\Gamma\rho^{\Gamma-2}\rho' - \frac{\omega^2\,(\omega^2-\omega_A^2)^2\,\rho_0^2\omega_A^2\,\tan^2\theta\,\rho'}{(\omega^2\,\rho_0-\omega_A^2\,\rho)^3}$$
$$+\Lambda\int\left\{\frac{4\pi G\rho_0}{\Lambda}+\frac{\omega^2}{2\,k^2}\left(1-\frac{\rho_0^2}{\rho^2}\right)+\frac{K\Gamma\left(\rho_0^{\Gamma-1}-\rho^{\Gamma-1}\right)}{\Gamma-1}\right.$$
$$\left.+\frac{\omega^2\,\tan^2\theta}{2\,k^2}\left(1-\frac{(\omega^2-\omega_A^2)^2\,\rho_0^2}{(\omega^2\,\rho_0-\omega_A^2\,\rho)^2}\right)\right\}d\chi = 4\,\pi\,G\int\rho\,d\chi. \tag{8.54}$$

This also (just like equation (8.51)), may be used in the determination of the higher order coefficients in the particle density. In fact a further integration is possible. The integrals do not present any difficulty as ρ is represented by exponentials and thus the integrals correspond to an algebraic operation. However, some terms (e.g. $4\pi G\rho_0$ and other constant terms) introduce χ itself after integration and a situation like in chapter 3 occurs.

Alternatively, one may differentiate equation (8.13) with respect to χ and then substitute (8.48) to determine an equation which can be used as well as equations (8.51) and (8.54), in the determination of the coefficients of the higher order quantity (the particle density):

$$\omega^2\,\rho_0^2\left(\frac{\rho'''}{\rho^3}-3\frac{\rho''\rho'}{\rho^4}-6\frac{\rho'\rho''}{\rho^4}+12\frac{\rho'^3}{\rho^5}\right)$$
$$-k^2 K\Gamma(\Gamma-2)\rho^{\Gamma-3}\rho'\left[\rho''+(\Gamma-2)\,\frac{\rho'^2}{\rho}\right]$$
$$-k^2 K\Gamma\rho^{\Gamma-2}\left[\rho'''-(\Gamma-2)\,\frac{\rho'^3}{\rho^2}+2\,(\Gamma-2)\,\frac{\rho'\rho''}{\rho}\right]$$
$$-\frac{\omega^2\,(\omega^2-\omega_A^2)^2\,\rho_0^2\,\omega_A^2\,\tan^2\theta}{(\omega^2\,\rho_0-\omega_A^2\,\rho)^3}\left(\rho'''+\frac{6\,\omega_A^2\,\rho'\rho''}{(\omega^2\,\rho_0-\omega_A^2\,\rho)}+\frac{3\,\omega_A^4\,\rho'^3}{(\omega^2\,\rho_0-\omega_A^2\,\rho)^2}\right)$$
$$-\frac{3\omega^2\,(\omega^2-\omega_A^2)^2\,\rho_0^2\,\omega_A^4\,\tan^2\theta\,\rho'}{(\omega^2\,\rho_0-\omega_A^2\,\rho)^4}\left(\rho''+\frac{3\,\omega_A^2\,\rho'^2}{(\omega^2\,\rho_0-\omega_A^2\,\rho)}\right)$$
$$+\Lambda\left\{\frac{\omega^2\,\rho_0^2}{k^2\,\rho^3}-K\Gamma\rho^{\Gamma-2}-\frac{\omega^2\,(\omega^2-\omega_A^2)^2\,\rho_0^2\,\omega_A^2\,\tan^2\theta}{k^2\,(\omega^2\,\rho_0-\omega_A^2\,\rho)^3}-\frac{4\,\pi\,G}{\Lambda}\right\}\rho'=0, \tag{8.55}$$

This seems more involved and uses third order differentials.

8.3.3.3 Nonlinear Fourier analysis

Further eliminations and reductions become rather complicated. Hence we introduce a Fourier analysis to algebraize the system. This introduces a hierarchy of coupled equations. We suppose that each quantity is represented by a Fourier series. E.g.

$$\rho = \rho_0 \sum_{j=0}^{\infty} \eta_j\, e^{ijx}, \qquad \varphi = \sum_{j=0}^{\infty} \varphi_j\, e^{ijx} \tag{8.56}$$

with η_j ($j = 0,\ 1,\ 2,\ \cdots$) constants and $\eta_0 = 1$. Identification of the coefficients in equation (8.13) yields:

$$\varphi_j = \frac{4\,\pi\, G\, \rho_0\, \eta_j}{\Lambda - j^2\, k^2}. \tag{8.57}$$

For $j = 0$ we recover equation (8.10) as $\eta_0 = 1$. The solution of the homogeneous part in equation (8.13) is left out here: as it is independent of ρ it seems rather meaningless. Including the homogeneous solution in φ leads to corresponding terms in v, which ultimately lead to an inconsistency in equation (8.16).

Equation (8.5) yields

$$p = K \rho_0^{\Gamma} \left(\sum_{j=0}^{\infty} \eta_j\, e^{ijx} \right)^{\Gamma} \tag{8.58}$$

or explicitly say to fourth order

$$\begin{aligned} p = K\rho_0^{\Gamma} \Bigg\{ & 1 + \Gamma\, \eta_1\, e^{ix} + \Gamma \left[\eta_2 + \frac{(\Gamma-1)\,\eta_1^2}{2} \right] e^{2ix} \\ & + \Gamma \left[\eta_3 + (\Gamma-1)\,\eta_1 \left(\eta_2 + \frac{(\Gamma-2)\,\eta_1^2}{6} \right) \right] e^{3ix} \\ & + \Gamma \left[\eta_4 + \frac{(\Gamma-1)}{2} \left(2\,\eta_1\eta_3 + \eta_2^2 + (\Gamma-2)\,\eta_1^2\eta_2 + \frac{(\Gamma-2)\,(\Gamma-3)\,\eta_1^4}{12} \right) \right] e^{4ix} \Bigg\} \end{aligned} \tag{8.59}$$

The values of η_j's may be obtained by substituting equation (8.56) (the first one) into either (8.51), (8.54) or (8.55).

8.4 Conclusion

In this chapter we studied the case where an infinity homogeneous gravitating medium with a cosmological term is pervaded by a homogeneous magnetic field. Our stability analysis in this case was split into two: incompressible and compressible cases.

For the case of incompressible Alfvén waves, the well-known dispersion relation of Alfvén waves (equation (8.23)) remained unchanged, even though the gravitation was included in the basic equations. Moreover, we see that arbitrary solutions satisfy equations (8.21), (8.22) and (8.23). This means that any motion in χ, with ω being given by (8.23), is allowed.

The analysis for the compressible case was the split into two: once with out gravitation and once with gravitation:
For the case of compressible Alfvén waves without gravitation we were able to reduce the the system of equations to an algebraic equation containing only ρ. Although from this equation we recovered the dispersion relation for Alfvén waves in a compressible medium as given by Chandrasekar [58], but we had zero convergence for higher order terms. Only constant solutions for the density are allowed: no motions based on the combined variable alone can take place. This contrasts not only with the incompressible case where arbitrary motions were allowed (however, satisfying equations (8.21), (8.22) and (8.23)), but also with our nonlinear Fourier analysis for plasmas and gravitating media studied in the previous chapters as well as in [46, 59, 60, 61, 62]. The case of sound waves (in their simplest form) on the other hand was similar and using our nonlinear analysis led to infinite coefficients and thus to zero convergence.

For the case of compressible gravito-Alfvén waves, we obtained the dispersion relation generalizing the dispersion relation of Chandrasekhar [9] for compressible Alfvén waves in an infinite homogeneous universe with Newtonian gravitation (made consistent by adding a cosmological term). In this case we had no zero convergence and hence higher order analysis was possible.

Chapter 9

Summary

General consideration(s). A **Fourier stability analysis** going to second order or at most fourth order only was developed by Callebaut [12] in the cases of hydrodynamics (liquid jet with surface tension), magnetohydrodynamics (plasma cylinder pervaded with uniform magnetic field), magnetogravito-dynamics (spiral arm of a galaxy pervaded by uniform magnetic field). Using the same basic method the nonlinear Fourier perturbation analysis for the case of infinite and homogeneous plasma media as well as gravitating media was elaborated in this thesis. Moreover, in the case of gravitation we adapted the stability method to construct nonhomogeneous equilibria and even evolution situations as possible models of the universe, however in a Newtonianlike version only. With the assistance of *mathematica* we were able to calculate a great number of terms. In some cases we were even able to derive analytic expressions for the higher order coefficients. Using these we managed to investigate the convergence of the results obtained. These results were confirmed using a graphical method, which is suitable too when there is no general analytical expression. Moreover, we indicated simple criteria which allow to predict bad convergence and thus when other nonlinear approaches should be used.

In **chapter 1** the general introduction was given. We considered the outline of the Fourier stability analysis developed by Callebaut. We then surveyed briefly the plasmas and gravitation subjects.

In **chapter 2** we used the nonlinear Fourier analysis developed by Callebaut [12] for an infinite homogeneous plasma calculating many higher order terms (computer algebra) and obtained in this way some analytic expressions which allowed to deal with the convergence. (a) *For cold plasma*: the maximum amplitude of the first order term is $1/e \approx 0.367879\cdots$ of the initial density, n_0, (i.e. at most 36.8% of n_0), otherwise the series diverges. If cosines are used instead of exponentials, then the maximum amplitude becomes $2/e \approx 0.735758\cdots$ (at most 73.6% of n_0) for the series to converge. (b) *For plasma with electron pressure*, the radius of convergence decreases as the ratio of $k^2 v_{s-}^2(1+\Gamma_-)/\omega_-^2$ increases (Γ_- is the polytropic exponent; ω_- is the plasma angular frequency for electrons; k is the wave number; v_{s-} is the sound velocity for the electrons).

The **graphical method** is the method which requires one to sketch the graphs of χ ($= \omega t + \boldsymbol{k}\cdot\boldsymbol{r}$) against say the density ρ (or n_-, the particle density, for the plasma case) for various values of the amplitude A (of the first order term) and N (the number of terms taken into account) with ρ_0 (the density at equilibrium) being scaled to one. For divergent series (i.e. when the series involves values of A which are greater or approximately equal to the maximum value of A) the graphs give some negative values in ρ, which is physically prohibited. But for the convergent series (i.e. when the series involves values of A which are less than the maximum A), all values of ρ (in the graphs) are always positive. Once the limit of convergence is exceeded the divergency occurs even when one uses few terms. However, around the limiting case (when A barely exceeds the maximum A) one sometimes needs a lot of terms to have a negative ρ. This graphical method has been applied in several other chapters too.

In **chapter 3** we again applied the Callebaut's nonlinear Fourier analysis to an infinite multi-species, homogeneous plasma calculating many higher order terms using *mathematica* and obtaining in this way some analytic expressions as before. The method used in this chapter is the improvement of the method discussed in the previous chapter. Hence we confirmed our previous results and add new ones. That is to say that in this chapter: (a) The results for cold plasma and for electron plasma given respectively in (a) and (b) above were confirmed. (b) Suggestions for experimental verification were made. (c) The cases where the method gives zero convergence were indicated. (d) Plasmas where both positive and negative ions may move, were considered in detail in particular the electron-positron plasmas (pulsar atmospheres).

The same results were obtained from nonlinear differential equation of second order, the one of the first order and the fully integrated one: as in chapter 2 the procedures were compared to see which one is the fastest.

In **chapter 4** the improved method was applied to an infinite multi-species, homogeneous plasma including the magnetic effects due to currents created by the motions of the charged particles. This implies therefore that we now use the fluid dynamic equations together with the full equations of Maxwell. As compared to the previous cases (plasma without the magnetic terms) where only longitudinal waves occurred, now we have two clear distinct cases: transversal and longitudinal waves. For the case of longitudinal waves, we obtained the same results, but more complete, as those obtained previously. In addition we are now able to write down explicitly the higher order terms for the current density and electric field (and to check a consistency relation). For the transversal waves, we recovered the dispersion relation for electromagnetic waves propagating in plasma with no external magnetic field. The particle density and pressure were found to be constant (not perturbed). For the case of velocity, electric field and current density, one has to fix one of the quantities, say electric field, in order to determine the other quantities.

In **chapter 5** we apply the improved nonlinear analysis developed previously to a (quasi-) infinite homogeneous gravitational medium in equilibrium. We use the Newtonian approximation with the exception that we introduce in the gravitational field equation, a cosmological constant, Λ, so as to make the equilibrium possible. Several higher order terms were calculated using *mathematica*. Three different ways (just as is the case in the previous chapters) were explored to handle the equations and to find out which one is best suited. The stability criterion of Jeans is confirmed in the limit when Λ approaches zero. This is quite satisfying as Jeans had derived his criterion from an inconsistent equilibrium and because the stability criterion is used not only for the universe but for stars and galaxies as well. The higher order terms allow corrections to the linearized theory; moreover using analytical extensions or graphical methods we are able to make statements on the convergence of the Fourier series. As we have used the polytropic law for the relation between pressure and density ($p = K\rho^{\Gamma}$) with an arbitrary polytropic exponent Γ the results are applicable equally well to the adiabatic case as to the isothermal one (when radiation should flat out the temperature differences) or any intermediate or alternative one. The results in this chapter were then compared to those obtained for plasmas. We may remark here that the interest in the cosmological term has been revived in particular due to the recent findings of the supernovae acting as standard candles to look back into the distant past of the universe.

In **chapter 6**, we again considered the medium infinite in all space dimensions and obeying the Newtonian law of gravitation to which a cosmological term is added. But now, as opposed to the analysis in chapter 5, we analyze the equilibria involving a varying gravitational potential and density. In this chapter we obtained a *non-uniform* solution: a specific cosinusoidal equilibrium for the polytropic exponent $\Gamma = 2$. Moreover this has another remarkable feature: its amplitude is arbitrary. This suggests that along the equilibria with varying amplitudes the stability is neutral or indifferent. A linearized perturbation analysis confirmed this view. This non-uniform equilibrium may be a first step towards the so-called tessellation of the universe:

observations indicate that the galaxies accumulate at certain (irregular) polyhedric walls and desert the interior of those polyhedra (cf. Jeans' criterion). However, using the suitable interpretation of the total pressure (gas pressure and radiation pressure) one may find dimensions of the order of the tessellation as well as of galaxies or clusters of galaxies.

In **chapter 7** we consided an infinite, nonuniform gravitating medium in motion (i.e. the initial velocity, v_0, is now different from zero). The cosmological constant is still necessary in order to avoid an empty universe. Although formally the present analysis of the evolution of a dynamic inhomogeneous turned out to be quite similar to a previous analysis, the interpretation is different. Moreover an infinity of wavenumbers, k_0, is now possible while in the previous chapter, with a strict interpretation of the equilibrium ($v_0 = 0$ and $v_0' = 0$), only one k_0 was allowed.

In **chapter 8** we once more study the infinite homogeneous medium using the Newtonian gravitation supplemented with a cosmological term in order to start with the correct equilibrium. The difference here (as compared to the previous case) is that, we now assume that our medium is pervaded by a homogeneous magnetic field (the (ionized) matter is supposed to be perfectly conducting). In this chapter we found three totally different situations with our method of combined variable:
a) Alfvén waves, incompressible medium. Full arbitrariness of the solutions satisfying the dispersion relation of Alfvén.
b) Alfvén waves, compressible nongravitating medium. Dispersion relation recovered but zero convergence for higher order terms.
c) Alfvén waves, compressible gravitating medium. Higher order analysis is possible.

The conclusions of this research are summarized in **chapter 9** and we give the Dutch translation of the summary (chapter 9) in **chapter 10**.

Chapter 10

Samenvatting

Algemene beschouwingen. Een Fourier stabiliteits analyse tot tweede orde, werd of hoogstens vierde orde, ontwikkeld door Callebaut [12] in het geval van hydrodynamica (vloeistof kolom met oppervlakte spanning), magnetodynamica (plasma cilinder doordrongen met een uniform magneetveld), magnetogravito-dynamica (spiraal arm van een galaxie doordrongen met een uniform magneetveld). Gebruik makend van dezelfde methode werd de niet lineaire Fourier storingsrekening in deze thesis uitgevoerd voor het geval van een oneindig uitgebreid homogeen plasma medium alsook voor gravitationele media. Bovendien werd de stabiliteitsmethode aangepast om niet homogene equilibria en zelfs evolutionele toestanden te construeren als mogelijke modellen van het universum, weliswaar slechts in een Newtoniaanse versie. Met behulp van *mathematica* kon een groot aantal termen berekend worden. In sommige gevallen slaagden we er zelfs in hieruit analytische uitdrukkingen af te leiden voor de hogere orde coëfficiënten. Hiermede lukte het om de convergentie van de resultaten na te gaan. Deze resultaten werden bevestigd door de grafische methode, welke ook kan gebruikt worden als er geen algemene analytische uitdrukking verkregen is. Bovendien werden eenvoudige criteria gegeven die slechte convergentie kunnen voorspellen en wanneer er dus andere niet lineaire benaderingen zouden moeten gebruikt worden.

In **hoofdstuk 1** werd een algemene inleiding gegeven. De Fourier stabiliteits analyse ontwikkeld door Callebaut werd geschetst. We maakten een kort overzicht van de plasma en gravitatie onderwerpen.

In **hoofdstuk 2** pasten we de niet lineaire Fourier analyse ontwikkeld door Callebaut [12] toe op een oneindig homogeen plasma en berekenden vele hogere orde termen (met computer algebra). Aldus verkregen we enige analytische uitdrukkingen die toelieten de convergentie te onderzoeken. (a) *Voor een koud plasma*: de maximale amplitude van de term van eerste orde is $1/e$ van de initiele dichtheid n_0 (d.w.z. hoogstens 36,8% van n_0); voor grotere amplitudes is de reeks divergent. Als cosinussen gebruikt worden i.p.v. exponentielen, dan wordt de maximale amplitude $2/e$ (dus hoogstens 73, 6% van n_0) opdat de reeks nog convergent zou zijn. (b) *Voor een plasma met elektronen druk* daalt de straal van convergentie als de verhouding $k^2V_{s-}^2(1+\Gamma_-)/\omega_-^2$ toeneemt (Γ_- is de polytrope exponent, ω_- is de plasma angulaire frekwentie voor elektronen; k is het golfgetal; v_{s-} is de geluidssnelheid voor de elektronen).

De **grafische methode** is die welke toelaat grafieken te schetsen van $\chi\,(=\omega t + \boldsymbol{k}\cdot\boldsymbol{r})$ tegen bv de dichtheid ρ (of n_-, de deeltjes dichtheid in het plasma geval) voor verscheidene waarden van de amplitude A (van de eerst orde term) en N (het aantal termen dat in acht genomen wordt) met ρ_0 (de evenwichtsdichtheid) als eenheid. Voor een divergente reeks (d.w.z. als de reeks werkt met waarden van A die groter zijn dan de maximale A) geven de grafieken enkele negatieve waarden voor ρ, wat fysisch niet toegelaten is. Voor een convergente reeks echter (d.w.z. wanneer de reeks werkt met waarden van A die kleiner zijn dan de maximale A), zijn alle waarden van ρ steeds positief in de grafieken. Eens de convergentie limiet overschreden is treedt de divergentie zelfs op bij gebruik van enkele termen. Maar naby het limiet geval (als A nauwelijks groter is dan de maximale A) heeft men soms een enorm aantal termen nodig vooraleer een negative ρ verkregen wordt. Deze grafische methode is toegepast in verscheidene andere hoofdstukken.

In **hoofdstuk 3** hebben we weer de niet lineaire Fourier analyse van Callebaut toegepast op een oneindig, multi-species homogeen plasma en talloze hogere order termen berekend met *mathematica* en aldus enkele analytische uitdrukkingen verkregen als voorheen. De methode is een verbetering van die van vorig hoofdstuk. Aldus werden de vorige resultaten bevestigd en enkele nieuwe toegevoegd. D.w.z.: (a) de resultaten voor het koud plasma en voor het elektronen plasma als in (a) en (b) hierboven gegeven werden bevestigd. (b) Suggesties voor experimentele verificaties werden gemaakt. (c) De gevallen waar de methode geen convergentie geeft werden aangeduid. (d) De gevallen waarin zowel positieve als negatieve ionen bewegen werden nu in meer detail beschouwd, in het bijzonder de elektron-positron plasma (pulsar atmosferen)

Dezelfde resultaten werden verkregen met de niet lineaire differentiaalvergelijking van tweede orde, met die van eerste orde en met de volledig geïntegreerde vergelijking: zoals in hoofdstuk 2 werden de procedures vergeleken om te zien welke het snelst tot resultaat leidt.

In **hoofdstuk 4** werd de verbeterde methode toegepast op een oneindig multi-species homogeen plasma waarbij de magnetische effecten veroorzaakt door de stromen gecreëerd door de bewegingen van de geladen deeltjes in acht genomen wordt. Dit betekent dat we de vergelijkingen van de fluidum dynamica beschouwen samen met de Maxwell vergelijkingen. Vergeleken met de voorgaande gevallen (plasmas zonder rekening te houden met magnetische termen), waar slechts longitudinale golven optraden, hebben we nu twee duidelijk onderscheiden gevallen: transversale en longitudinale golven. Voor het geval van de longitudinale golven verkregen we dezelfde, maar meer volledige, resultaten als voorheen. Bovendien kunnen we nu expliciet de hogere orde termen voor de stroomdichtheid en voor het elektrisch veld neerschrijven en de consistentie verifiëren. Voor de transversale golven vonden we de dispersie relatie terug voor elektromagnetische golven die zich voortplanten in een plasma zonder uitwendig magnetisch veld. De deeltjes dichtheid en de druk bleven constant, dus niet geperturbeerd. Voor het geval van de snelheid, het elektrisch veld en de stroomdichtheid kan men één grootheid kiezen, b.v. het elektrisch veld waaruit de andere grootheden dan volgen.

In **hoofdstuk 5** hebben we de verbeterde niet lineaire analyse toegepast op een (quasi) oneindig homogeen gravitationeel medium in evenwicht. We gebruiken de Newtoniaanse benadering, met de uitzondering dat we een kosmologische constante, Λ, invoeren in de gravitationele veldvergelijking om het evenwicht mogelijk te maken. Verscheidene hogere orde termen werden berekend met *mathematica*. Drie verschillende wegen (net zoals in voorgaande hoofdstukken) werden uitgewerkt voor de vergelijkingen om uit te zoeken welke de meest geschikte is. Het stabiliteitscriterium van Jeans is bevestigd in de limiet dat Λ naar nul nadert. Dit is zeer bevredigend vermits Jeans zijn criterium had afgeleid op grond van een inconsistent evenwicht en omdat het stabiliteitscriterium is gebruikt niet alleen voor het universum, maar ook voor sterren en galaxieën. De hogere order termen laten correcties toe op de lineaire theorie; bovendien, door gebruik te maken van analytische veralgemeningen of grafische methoden slaagden we er in uitspraken te maken over de convergentie van de Fourier reeks. Vermits we de polytrope wet gebruikten voor het verband tussen druk en dichtheid ($p = K\rho^{\Gamma}$) met een arbitraire polytrope exponent Γ zijn de resultaten evengoed toepasselijk op het adiabatische geval als op het isotherme geval (wanneer de straling de temperatuurverschillen zou uitvlakken) als op om 't even welke intermediair of alternatief geval. De resultaten van dit hoofdstuk werden vergeleken met die voor plasmas. Laten we nog opmerken dat de interesse voor de kosmologische term heraangewakkerd is door recente waarnemingen over supernovae die als "standaard kaarsen" toelaten terug te kijken in het verre verleden van het universum.

In **hoofdstuk 6** beschouwen we opnieuw een medium dat oneindig is in alle ruimtelijke dimensies en dat voldoet aan de Newtoniaanse gravitatiewet waaraan een kosmologische term is toegevoegd. Maar, in tegenstelling tot hoofdstuk 5, beschouwen we media met een veranderlijke potentiaal en dichtheid. In dit hoofdstuk verkregen we een *nietuniforme* oplossing: a welbepaald cosinusoïdaal evenwicht voor de polytrope exponent $\Gamma = 2$. Bovendien vertoont dit een andere merkwaardige eigenschap: de amplitude ervan is willekeurig. Dit suggereert dat langs de evenwichten met veranderlijke amplitude de stabiliteit neutraal is of onverschillig. Een gelineariseerde perturbatie analyse bevestigde dit gezichtspunt. Dit niet-uniforme evenwicht kan een eerste stap betekenen naar de zogenaamde tessellatie van het universum: observaties geven aan dat de galaxieën samenhopen op bepaalde (onregelmatige) polyedrische wanden en het inwendige van deze polyedra te verlaten (cf. Jeans criterium). Mits de prassende interpretatie van de rol van de totale druk (gasdruk en stralingsdruk) kennen zowel ofmetingen van de orde van de tesselatie als van galaxieën en clusters van g alaxieën verkregen woeden.

In **hoofdstuk 7** beschouwden we een oneindig, niet-uniform graviterend medium in beweging (d.w.z. de beginsnelheid, v_0, is nu verschillend van nul). De kosmologische constante is nog steeds nodig om een leeg universum te vermijden. Alhoewel de huidige analyse van de evolutie van een dynamisch inhomogeen universum formeel zeer gelijkaardig is aan de voorgaande analyses, is de interpretatie verschillend. Bovendien is nu een oneindig aantal golfgetallen, k_0, mogelijk, terwijl in voorgaand hoofdstuk, met een strikte interpretatie van het evenwicht ($v_0 = 0$ and $v_0' = 0$), slechts één k_0 toegelaten was.

In **hoofdstuk 8** bestuderen we nogmaals een oneindig homogeen medium op basis van de Newtoniaanse gravitatie waaraan een kosmologische term is toegevoegd om te starten met een correct evenwicht. Het verschil hier (in vergelijking met voorgaande gevallen) is dat we nu aannemen dat ons medium doordrongen is van een homogeen magnetisch veld (de (geïoniseerde) materie is ondersteld perfekt geleidend te zijn). In dit hoofdstuk vonden we drie totaal verschillende situaties d.m.v. onze methode van gecombineerde veranderlijke:

a) Alfvén golven, onsamendrukbaar medium: volledig arbitraire oplossingen, zodra aan de Alfvén dispersievergelijking voldaan is.

b) Alfvén golven, samendrukbaar niet gravitationeel medium: Dispersierelatie teruggevonden, maar nul convergentie voor de hogere orde termen.

c) Alfvén golven, samendrukbaar gravitationeel medium: de hogere order analyse is nu wel mogelijk.

De besluiten van dit onderzoek werden samengevat in het Engels in **hoofdstuk 9**. Nederlandse vertaling van de samenvatting is hier in **hoofdstuk 10** gegeven.

Appendix A

Fragile reduction procedure

Here we illustrate a procedure which reduces a system of equations to a fully integrated equation but produces results in tricky ways.

Consider equation (2.24) and rearrange after taking a square roots on its both sides. Then use the method of separation of variables to integrate the resulted equation:

$$\int \frac{dn_-}{(n_- - n_0)\, n_-^2} = C_2 \pm \int \sqrt{\frac{-\omega_-^2}{\omega^2 n_0^4}}\, d\chi. \tag{A.1}$$

The minus sign under the square root is harmless because we will use complex values for n_-. Integration gives

$$\frac{\ln (n_- - n_0)}{n_0^2} - \frac{\ln n_-}{n_0^2} + \frac{1}{n_0\, n_-} = C_2 \pm \frac{i\,\chi\,\omega_-}{\omega\, n_0^2}. \tag{A.2}$$

From the $\pm$ sign, we drop the minus sign for the reasons to be pointed out latter. This means that equation (A.2) now becomes

$$\boxed{\frac{1}{n_0^2} \ln \left(\frac{n_- - n_0}{n_-}\right) + \frac{1}{n_0\, n_-} - C_2 - \frac{i\chi\omega_-}{\omega n_0^2} = 0.} \tag{A.3}$$

However the determination of the constant C_2 requires care. Putting $n_- = n_0$ yields $-\infty$ for $\ln\left[(n_- - n_0)/n_-\right]$. Hence we have to include the perturbation as will be shown later.

Determination of $a_{-j\rho}$'s using equation (A.3)

1. We first fix a constant C_2:
   By substituting $n_- = n_0 + n_{-1}$ into equation (A.3) and linearizing one

gets

$$\frac{1}{n_0^2}\left[\ln\left(\frac{n_{-1}}{n_0}\right) - \frac{n_{-1}}{n_0} + 1\right] - C_2 - \frac{i\chi\omega_-}{\omega\, n_0^2} = 0. \quad \text{(A.4)}$$

Putting (2.25) into (A.4) and simplifying yields

$$\ln A - A\,e^{i\chi} + 1 - C_2 n_0^2 + i\chi\left(1 - \frac{\omega_-}{\omega}\right) = 0. \quad \text{(A.5)}$$

Equating the coefficients we have

$A = 0$ (But since $A \neq 0$ then we omit it) and

$$\ln A + 1 - C_2 n_0^2 + i\chi\left(1 - \frac{\omega_-}{\omega}\right) = 0,$$

which is an expression involving real and imaginary parts. The real part fixes C_2, i.e. $C_2 = \frac{\ln A}{n_0^2} + \frac{1}{n_0^2}$, and the imaginary part gives $\omega = \omega_-$ if and only if $\chi \neq 0$. But since the equation has to be satisfied for all χ this exception is irrelevant. Notice that if we would have maintained the $\pm$ in (A.3), then the dispersion relation would have been $\omega = \pm\omega_-$. Hence equation (A.3) becomes

$$\boxed{\ln\left(\frac{n_- - n_0}{n_- A}\right) + \frac{n_0}{n_-} - 1 - i\chi = 0.} \quad \text{(A.6)}$$

2. Plug equation (2.26) (up to the second order) into (A.6) to get

$$\ln\left(\frac{n_{-1} + n_{-2}}{n_0 + n_{-1} + n_{-2}}\right) - 1 + n_0\left(n_0 + n_{-1} + n_{-2}\right)^{-1} - \ln A - i\chi = 0.$$

The series expansion of this gives, after rejecting all terms involving orders higher than two and substituting for n_{-1} and n_{-2}, the following equation:

$$(a_{-2p} - 2)Ae^{i\chi} - \frac{1}{2}(a_{-2p}^2 + 4a_{-2p} - 3)A^2e^{2i\chi} = 0.$$

From the coefficient of $Ae^{i\chi}$ in this equation we get $a_{-2p} = 2$ and we reserve the coefficient of $A^2e^{2i\chi}$ for the determination of a_{-3p}. Following the similar procedure, one is able to determine all a_{-jp}'s confirming again the values given in table 2.1.

Note that if we would have maintained the minus sign in equation (A.3), we would have found a coefficient a_{-2p} which still depends on χ, and has to be rejected. This justifies our decision to drop that minus sign in equation (A.3).

Appendix B

Coefficients of the higher order terms

B.1 Coefficients of the higher order terms for the case of plasma with pressure of electrons

In this section we give some of the results obtained for particle density, velocity and potential with

$$\omega_\Omega = \left[(1+\Gamma_-)\,\varepsilon\, k^2\,\Gamma_-\,K_-\,n_0^{\Gamma_-}\right]/\left(e\,n_0\right)^2 = (1+\Gamma_-)\,k^2\,\Gamma_-\,\Lambda_{D-}^2$$

$$= k^2\,v_{s-}^2\,(1+\Gamma_-)/\omega_-^2 = (1+\Gamma_-)\left[(\omega/\omega_-)^2-1\right],$$

where $\Lambda_{D-}^2 = \left(\varepsilon\,K_-\,n_0^{\Gamma_-}\right)/\left(e\,n_0\right)^2$. We calculated the coefficients up to order 15, but give only the first five here. Next we give the expressions for $\Gamma_- = \gamma_- = 5/3$ up to order ten.

B.1.1 Coefficients in the particle density

$b_{-1\rho} = 1$

$b_{-2\rho} = 2 + \frac{2}{3}\omega_\Omega$

$b_{-3\rho} = \frac{9}{2} + \frac{3}{4}\omega_\Omega^2 + \left(\frac{27}{8} + \frac{3\Gamma_-}{16}\right)\omega_\Omega$

$b_{-4\rho} = \frac{32}{3} + \frac{28}{27}\omega_\Omega^3 + \omega_\Omega^2\left(6 + \frac{5\Gamma_-}{9}\right) + \omega_\Omega\left(\frac{194}{15} + \frac{11\Gamma_-}{9} + \frac{2\Gamma_-^2}{45}\right)$

$$b_{-5\rho} = \frac{625}{24} + \frac{2075}{1296}\omega_\Omega^4 + \omega_\Omega^3\left(\frac{2425}{216} + \frac{575\Gamma_-}{432}\right) + \omega_\Omega^2\left(\frac{18385}{576} + \frac{8975\Gamma_-}{1728} + \frac{1795\Gamma_-^2}{6912}\right)$$
$$+ \omega_\Omega\left(\frac{6395}{144} + \frac{265\Gamma_-}{48} + \frac{205\Gamma_-^2}{576} + \frac{5\Gamma_-^3}{576}\right)$$

$$b_{-6\rho} = 64.8 + 185.02\,\omega_\Omega + 201.23\,\omega_\Omega^2 + 105.34\,\omega_\Omega^3 + 26.74\,\omega_\Omega^4 + 2.65\,\omega_\Omega^5$$

$$b_{-7\rho} = 163.40 + 600.78\,\omega_\Omega + 871.06\,\omega_\Omega^2 + 645.09\,\omega_\Omega^3 + 259.52\,\omega_\Omega^4 + 54.10\,\omega_\Omega^5$$
$$+ 4.59\,\omega_\Omega^6$$

$$b_{-8\rho} = 416.10 + 1895.69\,\omega_\Omega + 3486.57\,\omega_\Omega^2 + 3398.82\,\omega_\Omega^3 + 1913.52\,\omega_\Omega^4 + 626.22\,\omega_\Omega^5$$
$$+ 110.85\,\omega_\Omega^6 + 8.22\,\omega_\Omega^7$$

$$b_{-9\rho} = 1067.63 + 5858.22\,\omega_\Omega + 13199.1\,\omega_\Omega^2 + 16163.5\,\omega_\Omega^3 + 11876.9\,\omega_\Omega^4 + 5398.85\,\omega_\Omega^5$$
$$+ 1490.28\,\omega_\Omega^6 + 229.31\,\omega_\Omega^7 + 15.11\,\omega_\Omega^8$$

$$b_{-10\rho} = 2755.73 + 17819.6\,\omega_\Omega + 47925.3\,\omega_\Omega^2 + 71340.9\,\omega_\Omega^3 + 65403.9\,\omega_\Omega^4$$
$$+ 38567.1\,\omega_\Omega^5 + 14706.3\,\omega_\Omega^6 + 3511.42\,\omega_\Omega^7 + 477.97\,\omega_\Omega^8 + 28.34\,\omega_\Omega^9$$

B.1.2 Coefficients in the velocity (in unit $-\omega/k$)

$$b_{-1v} = 1$$

$$b_{-2v} = 1 + \frac{2\omega_\Omega}{3}$$

$$b_{-3v} = \frac{3}{2} + \left(\frac{49}{24} + \frac{3\,\Gamma_-}{16}\right)\omega_\Omega + \frac{3\,\omega_\Omega^2}{4}$$

$$b_{-4v} = \frac{8}{3} + \left(\frac{331}{60} + \frac{61\,\Gamma_-}{72} + \frac{2\,\Gamma_-^2}{45}\right)\omega_\Omega + \left(\frac{73}{18} + \frac{5\,\Gamma_-}{9}\right)\omega_\Omega^2 + \frac{28\,\omega_\Omega^3}{27}$$

$$b_{-5v} = \frac{125}{24} + \left(\frac{10441}{720} + \frac{26\,\Gamma_-}{9} + \frac{769\,\Gamma_-^2}{2880} + \frac{5\,\Gamma_-^3}{576}\right)\omega_\Omega + \left(\frac{9217}{576} + \frac{6623\,\Gamma_-}{1728} + \frac{1795\,\Gamma_-^2}{6912}\right)\omega_\Omega^2$$
$$+ \left(\frac{587}{72} + \frac{575\,\Gamma_-}{432}\right)\omega_\Omega^3 + \frac{2075\,\omega_\Omega^4}{1296}$$

$$b_{-6v} = 10.8 + 56.0669\,\omega_\Omega + 91.9886\,\omega_\Omega^2 + 65.9411\,\omega_\Omega^3 + 21.591\,\omega_\Omega^4 + 2.64722\,\omega_\Omega^5$$

$$b_{-7v} = 23.3431 + 154.71\,\omega_\Omega + 336.154\,\omega_\Omega^2 + 339.183\,\omega_\Omega^3 + 175.311\,\omega_\Omega^4$$

$$+\,45.1131\,\omega_\Omega^5 + 4.58598\,\omega_\Omega^6$$

$$b_{-8v} = 52.0127+424.258\,\omega_\Omega+1163.26\,\omega_\Omega^2+1538.65\,\omega_\Omega^3+1109.31\,\omega_\Omega^4+446.966\,\omega_\Omega^5 + 94.6671\,\omega_\Omega^6 + 8.21773\,\omega_\Omega^7$$

$$b_{-9v} = 118.625+1159.02\,\omega_\Omega+3876.32\,\omega_\Omega^2+6418.97\,\omega_\Omega^3+6023.44\,\omega_\Omega^4+3363.48\,\omega_\Omega^5 + 1108.87\,\omega_\Omega^6 + 199.472\,\omega_\Omega^7 + 15.1074\,\omega_\Omega^8$$

$$b_{-10v} = 275.573 + 3158.58\,\omega_\Omega + 12564.5\,\omega_\Omega^2 + 25218.5\,\omega_\Omega^3 + 29456.1\,\omega_\Omega^4 + 21295.7\,\omega_\Omega^5 + 9682.66\,\omega_\Omega^6 + 2699.08\,\omega_\Omega^7 + 421.865\,\omega_\Omega^8 + 28.3357\,\omega_\Omega^9$$

B.1.3 Coefficients in the potential (in unit $-(en_0)/(k^2\varepsilon)$)

$$b_{-1\varphi} = 1$$

$$b_{-2\varphi} = \tfrac{1}{2} + \tfrac{\omega_\Omega}{6}$$

$$b_{-3\varphi} = \tfrac{1}{2} + \left(\tfrac{3}{8} + \tfrac{\Gamma_-}{48}\right)\omega_\Omega + \tfrac{\omega_\Omega^2}{12}$$

$$b_{-4\varphi} = \tfrac{2}{3} + \left(\tfrac{97}{120} + \tfrac{11\,\Gamma_-}{144} + \tfrac{\Gamma_-^2}{360}\right)\omega_\Omega + \left(\tfrac{3}{8} + \tfrac{5\,\Gamma_-}{144}\right)\omega_\Omega^2 + \tfrac{7\,\omega_\Omega^3}{108}$$

$$b_{-5\varphi} = \tfrac{25}{24} + \left(\tfrac{1279}{720} + \tfrac{53\,\Gamma_-}{240} + \tfrac{41\,\Gamma_-^2}{2880} + \tfrac{\Gamma_-^3}{2880}\right)\omega_\Omega + \left(\tfrac{3677}{2880} + \tfrac{359\,\Gamma_-}{1728} + \tfrac{359\,\Gamma_-^2}{34560}\right)\omega_\Omega^2 + \left(\tfrac{97}{216} + \tfrac{23\,\Gamma_-}{432}\right)\omega_\Omega^3 + \tfrac{83\,\omega_\Omega^4}{1296}$$

$$b_{-6\varphi} = 1.8 + 5.14\,\omega_\Omega + 5.59\,\omega_\Omega^2 + 2.93\,\omega_\Omega^3 + 0.74\,\omega_\Omega^4 + 0.07\,\omega_\Omega^5$$

$$b_{-7\varphi} = 3.34 + 12.26\,\omega_\Omega + 17.78\,\omega_\Omega^2 + 13.17\,\omega_\Omega^3 + 5.3\,\omega_\Omega^4 + 1.10\,\omega_\Omega^5 + 0.09\,\omega_\Omega^6$$

$$b_{-8\varphi} = 6.50 + 29.62\,\omega_\Omega + 54.48\,\omega_\Omega^2 + 53.11\,\omega_\Omega^3 + 29.9\,\omega_\Omega^4 + 9.79\,\omega_\Omega^5 + 1.73\,\omega_\Omega^6 + 0.13\,\omega_\Omega^7$$

$$b_{-9\varphi} = 13.18 + 72.32\,\omega_\Omega + 162.95\,\omega_\Omega^2 + 199.55\,\omega_\Omega^3 + 146.63\,\omega_\Omega^4 + 66.65\,\omega_\Omega^5 + 18.4\,\omega_\Omega^6 + 2.83\,\omega_\Omega^7 + 0.19\,\omega_\Omega^8$$

$$b_{-10\varphi} = 27.56 + 178.2\,\omega_\Omega + 479.25\,\omega_\Omega^2 + 713.41\,\omega_\Omega^3 + 654.04\,\omega_\Omega^4 + 385.67\,\omega_\Omega^5$$
$$+ 147.06\,\omega_\Omega^6 + 35.11\,\omega_\Omega^7 + 4.78\,\omega_\Omega^8 + 0.28\,\omega_\Omega^9$$

B.2 The determined high order coefficients for two component cold plasma

B.2.1 Particle density

Here we give the values of the simultaneously determined coefficients for the particle density expressions when both electrons and ions in the plasma are oscillating. We take $c_{-1\rho} = 1$ and assume that $T_+ \approx 0 \approx T_-$ implying that $v_{s-} \approx 0 \approx v_{s+}$ and $\omega^2 = \omega_-^2 + \omega_+^2$ where $\omega_\pm^2 = (e^2 n_0)/(\varepsilon m_\pm)$. Thus the coefficients in the first five orders are

$c_{-1\rho} = 1$ and $c_{+1\rho} = -\frac{\omega_+^2}{\omega_-^2}$,

$c_{-2\rho} = \frac{4\omega_-^2 - \omega_+^2}{2\omega_-^2}$ and $c_{+2\rho} = \frac{4\omega_+^4 - \omega_-^2\omega_+^2}{2\omega_-^4}$,

$c_{-3\rho} = \frac{9}{2} - \frac{35\,\omega_+^2}{16\,\omega_-^2} + \frac{\omega_+^4}{2\,\omega_-^4}$ and $c_{+3\rho} = \frac{-\omega_+^2}{2\,\omega_-^2} + \frac{35\,\omega_+^4}{16\,\omega_-^4} - \frac{9\,\omega_+^6}{2\,\omega_-^6}$,

$c_{-4\rho} = \frac{32}{3} - \frac{23\,\omega_+^2}{3\,\omega_-^2} + \frac{143\,\omega_+^4}{48\,\omega_-^4} - \frac{2\,\omega_+^6}{3\,\omega_-^6}$ and $c_{+4\rho} = \frac{-2\,\omega_+^2}{3\,\omega_-^2} + \frac{143\,\omega_+^4}{48\,\omega_-^4} - \frac{23\,\omega_+^6}{3\,\omega_-^6} + \frac{32\,\omega_+^8}{3\,\omega_-^8}$,

$c_{-5\rho} = \frac{625}{24} - \frac{2375\omega_+^2}{96\omega_-^2} + \frac{10015\omega_+^4}{768\omega_-^4} - \frac{455\omega_+^6}{96\omega_-^6} + \frac{25\omega_+^8}{24\omega_-^8}$ and

$c_{+5\rho} = \frac{-25\omega_+^2}{24\omega_-^2} + \frac{455\omega_+^4}{96\omega_-^4} - \frac{10015\omega_+^6}{768\omega_-^6} + \frac{2375\omega_+^8}{96\omega_-^8} - \frac{625\omega_+^{10}}{24\omega_-^{10}}$.

B.2.2 Velocity (in unit (ω/k))

Here we have the following coefficients for the velocity

$$c_{\pm 1v} = -c_{\pm 1\rho}$$

$$c_{\pm 2v} = c_{\pm 1\rho}^2 - c_{\pm 2\rho}$$

$$c_{\pm 3v} = 2c_{\pm 1\rho}c_{\pm 2\rho} - c_{\pm 1\rho}^3 - c_{\pm 3\rho}$$

$$c_{\pm 4v} = c_{\pm 1\rho}^4 - 3c_{\pm 1\rho}^2 c_{\pm 2\rho} + c_{\pm 2\rho}^2 + 2c_{\pm 1\rho}c_{\pm 3\rho} - c_{\pm 4\rho}$$

$$c_{\pm 5v} = 4c_{\pm 1\rho}^3 c_{\pm 2\rho} - c_{\pm 1\rho}^5 - 3c_{\pm 1\rho}c_{\pm 2\rho}^2 - 3c_{\pm 1\rho}^2 c_{\pm 3\rho} + 2c_{\pm 2\rho}c_{\pm 3\rho} + 2c_{\pm 1\rho}c_{\pm 4\rho} - c_{\pm 5\rho}$$

B.2.3 Potential (in unit $(e\, n_0)/(k^2\, \varepsilon)$)

The coefficients for the potential can easily be determined from the following formula

$$c_{j\varphi} = \left(c_{+j\rho} - c_{-j\rho}\right)/j^2 \tag{B.1}$$

with $j = 1, 2, \cdots, 5$. The $c_{\pm j\rho}$'s are given in appendix B.2.1.

B.2.4 Electron-positron plasma without pressure term

The higher orders coefficients for the electron-positron plasma (cold plasma case) are determined by putting $\omega_+ = \omega_-$ into each of the the above coefficients. For example for the case of the particle density the first five coefficients will become:
$c_{\pm 1\rho} = \mp 1, \quad c_{\pm 2\rho} = \frac{3}{2}, \quad c_{\pm 3\rho} = \mp\frac{45}{16}, \quad c_{\pm 4\rho} = \frac{85}{16}, \quad c_{\pm 5\rho} = \mp\frac{2725}{256}.$

B.3 The coefficients for the two component plasma with pressure

In this appendix we consider the case when both ions and electrons in the plasma have time to move. We give here only the coefficients for the first and second order terms since other higher orders are very long. The coefficients are as follows:

(a) The coefficient for the particle density

$c_{-1\rho} = 1$

$$c_{+1\rho} = \frac{k^2\left(v_{s-}^2 - v_{s+}^2\right) + \omega_-^2 - \omega_+^2 \mp \sqrt{4\,\omega_-^2\,\omega_+^2 + \left[k^2\left(v_{s-}^2 - v_{s+}^2\right) + \omega_-^2 - \omega_+^2\right]^2}}{2\,\omega_-^2}. \tag{B.2}$$

$$c_{\pm 2\rho} = \frac{\beta_{\pm 1\rho} + \beta_{\pm 2\rho}}{\Phi}, \tag{B.3}$$

where

$$\beta_{\pm 1\rho} = 4\left(2\,k^2\,v_{s\mp}^2 + \omega_\mp^2 - 2\,\omega^2\right)\omega_\pm^2\,c_{+1\rho} - \left(\omega^2 + k^2\,v_{s\mp}^2\,\Gamma_\mp + 2\,\omega_\mp^2\right)\omega_\pm^2 c_{\mp 1\rho},$$

$$\beta_{\pm 2\rho} = \left(4\,\omega^2 - 4\,k^2\,v_{s\mp}^2 - \omega_\mp^2\right)\left(\omega^2 + k^2\,v_{s\pm}^2\,\Gamma_\pm + 2\,\omega_\pm^2\right)c_{\pm 1\rho}^2,$$

$$\Phi = 2\left[\left(\omega^2 - k^2 v_{s+}^2\right)\left(4\,\omega^2 - 4\,k^2 v_{s-}^2 - \omega_-^2\right) - \left(\omega^2 - k^2 v_{s-}^2\right)\omega_+^2\right].$$

Parameters ω^2 and $c_{+1\rho}$ are given by equations (3.26) and (B.2) respectively.

(b) The coefficients for the velocity (in unit (ω/k))

The expressions for the first and second order terms in velocity are determined by using equation (3.18). Thus we have

$$c_{\pm 1v} = -c_{\pm 1\rho} \tag{B.4}$$

and

$$c_{\pm 2v} = c_{\pm 1\rho}^2 - c_{\pm 2\rho}, \tag{B.5}$$

where $c_{+1\rho}$ and $c_{\pm 2\rho}$ are given by equations(B.2) and (B.3) respectively and $c_{-1\rho} = 1$.

(c) The coefficients for the potential (in unit $(e\,n_0)/(k^2\,\varepsilon)$)

The expressions for the first and second order terms in potential are determined from equation (3.21). Hence we have

$$c_{1\varphi} = c_{+1\rho} - c_{-1\rho} \tag{B.6}$$

and

$$c_{2\varphi} = \frac{c_{+2\rho} - c_{-2\rho}}{4}. \tag{B.7}$$

with $c_{-1\rho} = 1$ and $c_{+1\rho}$ and $c_{\pm 2\rho}$ being given by equations(B.2) and (B.3) respectively.

B.4 Higher order coefficients for the case of electron-positron plasma

In table 3.3 we gave the times required to determine the coefficients in the perturbed particle density for the case of electron-positron plasma when equations (3.22), (3.23) or (3.24) with $a_{\pm j\rho}$ being replaced by $cc_{\pm j\rho}$ and $cc_{-1\rho} = 1$ (using cc as a notation) were used. Here we will only write down the first five expressions for the determined coefficients.

We first wrote the *mathematica* programs which assisted us in the determination of the coefficients. For the case of the first order, we assumed that $cc_{-1\rho} = 1$ obtained simultaneously:

$$c_{+1\rho} = \frac{\omega_-^2 + k^2\, v_{s-}^2 - \omega^2}{\omega_-^2} \quad \text{and} \quad \omega = \pm\sqrt{\frac{\omega_-^2\ (c_{+1\rho} - 1) + k^2\, v_{s-}^2\, c_{+1\rho}}{c_{+1\rho}}}.$$

Putting the expression for $c_{+1\rho}$ into that of ω one obtains the dispersion relations: $\omega^2 = 2\omega_-^2 + k^2\, v_{s-}^2$ and $\omega^2 = k^2\, v_{s-}^2$, which are nothing other than the dispersion relations of Langmuir and sound waves respectively. We then substituted $\omega^2 = 2\omega_-^2 + k^2\, v_{s-}^2$ into equations (3.22), (3.23) and (3.24) and hence obtain the remaining higher order coefficients. The first five coefficients after rearrangement with $\omega_\Omega = (1 + \Gamma_-) k^2 v_{s-}^2 / \omega_-^2$, are given by:

$$cc_{\pm 1\rho} = \mp 1$$

$$cc_{\pm 2\rho} = (3/2) + (\omega_\Omega/4)$$

$$cc_{\pm 3\rho} = \mp\ \{(45/16) + [3\ (12 + \Gamma_-)\ \omega_\Omega/32] + (9\,\omega_\Omega^2/64)\}$$

$$\begin{aligned} cc_{\pm 4\rho} = {} & (85/16) + \left[\left(333 + 43\,\Gamma_- + 2\,{\Gamma_-}^2\right)\ \omega_\Omega/96\right] + [(57 + 7\,\Gamma_-)\ {\omega_\Omega}^2/64] \\ & + (11\,{\omega_\Omega}^3/128) \end{aligned}$$

$$\begin{aligned} cc_{\pm 5\rho} = {} & \mp\ \left\{(2725/256) + \left[5\ \left(4560 + 757\,\Gamma_- + 60\,{\Gamma_-}^2 + 2\,{\Gamma_-}^3\right)\ \omega_\Omega/2304\right]\right. \\ & + \left[25\ \left(486 + 104\,\Gamma_- + 7\,{\Gamma_-}^2\right)\ {\omega_\Omega}^2/3072\right] \\ & \left. + [25\ (32 + 5\,\Gamma_-)\ {\omega_\Omega}^3/1024] + (775\,{\omega_\Omega}^4/12288)\right\} \end{aligned}$$

B.5 Coefficients of the higher order terms for the case of uniform gravitational medium with cosmological constant

Here we give the determined (with the assistance of *mathematica* program of course) coefficients in the expressions of density, velocity and potential up to order ten. For simplicity, in each of the following sections, we just give expressions for the first four coefficients and for the remaining coefficients, we give the appropriate expressions after substituting $\Lambda = 0$ and $\Gamma = 5/3$.

The parameters k_J and ω_β, which appear in the coefficients, may be defined respectively as:

$$k_J = 2\sqrt{\frac{G\,\pi\,\rho_{g0}^{2-\Gamma}}{K\,\Gamma}} \quad \text{and} \quad \omega_\beta = \frac{(1+\Gamma)\,(k^2-\Lambda)}{{k_J}^2}.$$

B.5.1 Coefficients in the density expression

Using equations (5.22) and (5.23) in either equation (5.17), (5.19) or (5.21) yield the following coefficients:

$$a_{2g} = 2 - \frac{\Lambda}{2\,k^2} - \left(\frac{2}{3} - \frac{\Lambda}{6\,k^2}\right)\omega_\beta$$

$$a_{3g} = \frac{9}{2} - \frac{35\,\Lambda}{16\,k^2} + \frac{3\,\Lambda^2}{16\,k^4} - \left(\frac{27}{8} + \frac{3\,\Gamma}{16} - \frac{3\,\Lambda}{2\,k^2} - \frac{\Gamma\,\Lambda}{48\,k^2} + \frac{\Lambda^2}{8\,k^4}\right)\omega_\beta + \left(\frac{3}{4} - \frac{13\,\Lambda}{48\,k^2} + \frac{\Lambda^2}{48\,k^4}\right)\omega_\beta^2$$

$$\begin{aligned} a_{4g} = {} & \frac{32}{3} - \frac{23\,\Lambda}{3\,k^2} + \frac{23\,\Lambda^2}{16\,k^4} - \frac{\Lambda^3}{16\,k^6} - \left(\frac{194}{15} + \frac{11\,\Gamma}{9} + \frac{2\,\Gamma^2}{45} - \frac{339\,\Lambda}{40\,k^2} - \frac{59\,\Gamma\,\Lambda}{144\,k^2}\right. \\ & \left. - \frac{\Gamma^2\,\Lambda}{360\,k^2} + \frac{71\,\Lambda^2}{48\,k^4} + \frac{\Gamma\,\Lambda^2}{48\,k^4} - \frac{\Lambda^3}{16\,k^6}\right)\omega_\beta \\ & + \left(6 + \frac{5\,\Gamma}{9} - \frac{27\,\Lambda}{8\,k^2} - \frac{7\,\Gamma\,\Lambda}{48\,k^2} + \frac{25\,\Lambda^2}{48\,k^4} + \frac{\Gamma\,\Lambda^2}{144\,k^4} - \frac{\Lambda^3}{48\,k^6}\right)\omega_\beta^2 \\ & - \left(\frac{28}{27} - \frac{17\,\Lambda}{36\,k^2} + \frac{\Lambda^2}{16\,k^4} - \frac{\Lambda^3}{432\,k^6}\right)\omega_\beta^3 \end{aligned}$$

If $\Lambda = 0$ and $\Gamma = \frac{5}{3}$ then

$$a_{5g} = 26.0417 - 54.6399\,\omega_\beta + 41.2962\,\omega_\beta^2 - 13.4452\,\omega_\beta^3 + 1.60108\,\omega_\beta^4$$

$$a_{6g} = 64.8 - 185.017\,\omega_\beta + 201.23\,\omega_\beta^2 - 105.338\,\omega_\beta^3 + 26.7384\,\omega_\beta^4 - 2.64722\,\omega_\beta^5$$

$$a_{7g} = 163.401 - 600.783\,\omega_\beta + 871.06\,\omega_\beta^2 - 645.093\,\omega_\beta^3 + 259.518\,\omega_\beta^4$$

$$-54.0979\,\omega_\beta^5 + 4.58598\,\omega_\beta^6$$

$$a_{8g} = 416.102 - 1895.69\,\omega_\beta + 3486.57\,\omega_\beta^2 - 3398.82\,\omega_\beta^3 + 1913.52\,\omega_\beta^4$$
$$-626.223\,\omega_\beta^5 + 110.846\,\omega_\beta^6 - 8.21773\,\omega_\beta^7$$

$$a_{9g} = 1067.63 - 5858.22\,\omega_\beta + 13199.1\,\omega_\beta^2 - 16163.5\,\omega_\beta^3 + 11876.9\,\omega_\beta^4$$
$$-5398.85\,\omega_\beta^5 + 1490.28\,\omega_\beta^6 - 229.313\,\omega_\beta^7 + 15.1074\,\omega_\beta^8$$

$$a_{10g} = 2755.73 - 17819.6\,\omega_\beta + 47925.3\,\omega_\beta^2 - 71340.9\,\omega_\beta^3 + 65403.9\,\omega_\beta^4$$
$$-38567.1\,\omega_\beta^5 + 14706.3\,\omega_\beta^6 - 3511.42\,\omega_\beta^7 + 477.97\,\omega_\beta^8 - 28.3357\,\omega_\beta^9$$

B.5.2 Coefficients (in unit ω/k) for the velocity expression

In view of equation (5.22), we put the velocity into the form $v = (k/k)\sum_{j=1}^{N} b_{jg} A^j \exp(ij\chi)$. Thus using this, in conjunction with the obtained values of the density, in equation (5.14) we get:

$$b_{1g} = -1, \qquad b_{2g} = -1 + \frac{\Lambda}{2\,k^2} + \left(\frac{2}{3} - \frac{\Lambda}{6\,k^2}\right)\omega_\beta,$$

$$b_{3g} = -\left(\frac{3}{2}\right) + \frac{19\,\Lambda}{16\,k^2} - \frac{3\,\Lambda^2}{16\,k^4} + \left(\frac{49}{24} + \frac{3\,\Gamma}{16} - \frac{\left(\frac{7}{6} + \frac{\Gamma}{48}\right)\Lambda}{k^2} + \frac{\Lambda^2}{8\,k^4}\right)\omega_\beta -$$
$$\left(\frac{3}{4} - \frac{13\,\Lambda}{48\,k^2} + \frac{\Lambda^2}{48\,k^4}\right)\omega_\beta^2,$$

$$b_{4g} = -\frac{8}{3} + \frac{67\,\Lambda}{24\,k^2} - \frac{13\,\Lambda^2}{16\,k^4} + \frac{\Lambda^3}{16\,k^6} + \left[\frac{331}{60} + \frac{61\,\Gamma}{72} + \frac{2\,\Gamma^2}{45}\right.$$
$$\left. - \left(\frac{557}{120} + \frac{53\,\Gamma}{144} + \frac{\Gamma^2}{360}\right)\frac{\Lambda}{k^2} + \left(\frac{17}{16} + \frac{\Gamma}{48}\right)\frac{\Lambda^2}{k^4} - \frac{\Lambda^3}{16\,k^6}\right]\omega_\beta$$
$$-\left(\frac{73}{18} + \frac{5\,\Gamma}{9} - \left(\frac{47}{18} + \frac{7\,\Gamma}{48}\right)\frac{\Lambda}{k^2} + \frac{65\,\Lambda^2}{144\,k^4} + \frac{\Gamma\,\Lambda^2}{144\,k^4} - \frac{\Lambda^3}{48\,k^6}\right)\omega_\beta^2 +$$

$$\left(\frac{28}{27}-\frac{17\,\Lambda}{36\,k^2}+\frac{\Lambda^2}{16\,k^4}-\frac{\Lambda^3}{432\,k^6}\right)\,\omega_\beta^3.$$

Putting $\Lambda = 0$ and $\Gamma = \frac{5}{3}$ we have

$$b_{5g} = -5.20833 + 20.0981\,\omega_\beta - 23.111\,\omega_\beta^2 + 10.3711\,\omega_\beta^3 - 1.60108\,\omega_\beta^4,$$

$$b_{6g} = -10.8 + 56.0669\,\omega_\beta - 91.9886\,\omega_\beta^2 + 65.9411\,\omega_\beta^3 - 21.591\,\omega_\beta^4 + 2.64722\,\omega_\beta^5,$$

$$b_{7g} = -23.3431 + 154.71\,\omega_\beta - 336.154\,\omega_\beta^2 + 339.183\,\omega_\beta^3 - 175.311\,\omega_\beta^4 \\ +45.1131\,\omega_\beta^5 - 4.58598\,\omega_\beta^6,$$

$$b_{8g} = -52.0127 + 424.258\,\omega_\beta - 1163.26\,\omega_\beta^2 + 1538.65\,\omega_\beta^3 - 1109.31\,\omega_\beta^4 \\ +446.966\,\omega_\beta^5 - 94.6671\,\omega_\beta^6 + 8.21773\,\omega_\beta^7,$$

$$b_{9g} = -118.625 + 1159.02\,\omega_\beta - 3876.32\,\omega_\beta^2 + 6418.97\,\omega_\beta^3 - 6023.44\,\omega_\beta^4 \\ +3363.48\,\omega_\beta^5 - 1108.87\,\omega_\beta^6 + 199.472\,\omega_\beta^7 - 15.1074\,\omega_\beta^8,$$

$$b_{10g} = -275.573 + 3158.58\,\omega_\beta - 12564.5\,\omega_\beta^2 + 25218.5\,\omega_\beta^3 - 29456.1\,\omega_\beta^4 \\ +21295.7\,\omega_\beta^5 - 9682.66\,\omega_\beta^6 + 2699.08\,\omega_\beta^7 - 421.865\,\omega_\beta^8 + 28.3357\,\omega_\beta^9.$$

B.5.3 Coefficients (in unit $k_J^2\, v_s^2/k^2$) for the potential expression

We put $\varphi = \sum_{j=1}^{N} c_{jg} A^j \exp(ij\chi)$, and use it with the determined values of the densities in equation (5.16). This process gives

$$c_{0g} = \frac{k^2}{\Lambda} \qquad c_{1g} = \frac{k^2}{\Lambda - k^2} \qquad c_{2g} = -\frac{1}{2} + \frac{\omega_\beta}{6}$$

$$c_{3g} = -\frac{1}{2} + \frac{3\,\Lambda}{16\,k^2} + \left(\frac{3}{8} + \frac{\Gamma}{48} - \frac{\Lambda}{8\,k^2}\right)\,\omega_\beta - \left(\frac{1}{12} - \frac{\Lambda}{48\,k^2}\right)\,\omega_\beta^2$$

$$c_{4g} = -\frac{2}{3} + \frac{7\,\Lambda}{16\,k^2} - \frac{\Lambda^2}{16\,k^4} + \left(\frac{97}{120} + \frac{11\,\Gamma}{144} + \frac{\Gamma^2}{360} - \frac{23\,\Lambda}{48\,k^2} - \frac{\Gamma\,\Lambda}{48\,k^2} + \frac{\Lambda^2}{16\,k^4}\right)\omega_\beta$$

$$-\left(\frac{3}{8} + \frac{5\,\Gamma}{144} - \frac{3\,\Lambda}{16\,k^2} - \frac{\Gamma\,\Lambda}{144\,k^2} + \frac{\Lambda^2}{48\,k^4}\right)\omega_\beta^2 + \left(\frac{7}{108} - \frac{11\,\Lambda}{432\,k^2} + \frac{\Lambda^2}{432\,k^4}\right)\omega_\beta^3$$

If $\Lambda = 0$ and $\Gamma = \frac{5}{3}$ we obtain

$$c_{5g} = -1.04 + 2.19\,\omega_\beta - 1.65\,\omega_\beta^2 + 0.54\,\omega_\beta^3 - 0.064\,\omega_\beta^4$$

$$c_{6g} = -1.8 + 5.14\,\omega_\beta - 5.59\,\omega_\beta^2 + 2.93\,\omega_\beta^3 - 0.74\,\omega_\beta^4 + 0.074\,\omega_\beta^5$$

$$c_{7g} = -3.33 + 12.26\,\omega_\beta - 17.78\,\omega_\beta^2 + 13.17\,\omega_\beta^3 - 5.3\,\omega_\beta^4 + 1.104\,\omega_\beta^5 - 0.094\,\omega_\beta^6$$

$$c_{8g} = -6.501 + 29.62\,\omega_\beta - 54.48\,\omega_\beta^2 + 53.11\,\omega_\beta^3 - 29.899\,\omega_\beta^4 + 9.78\,\omega_\beta^5$$
$$-1.73\,\omega_\beta^6 + 0.13\,\omega_\beta^7$$

$$c_{9g} = -13.18 + 72.32\,\omega_\beta - 162.95\,\omega_\beta^2 + 199.55\,\omega_\beta^3 - 146.63\,\omega_\beta^4 + 66.65\,\omega_\beta^5$$
$$-18.399\,\omega_\beta^6 + 2.83\,\omega_\beta^7 - 0.19\,\omega_\beta^8$$

$$c_{10g} = -27.56 + 178.196\,\omega_\beta - 479.25\,\omega_\beta^2 + 713.41\,\omega_\beta^3 - 654.04\,\omega_\beta^4 + 385.67\,\omega_\beta^5$$
$$-147.06\,\omega_\beta^6 + 35.11\,\omega_\beta^7 - 4.78\,\omega_\beta^8 + 0.28\,\omega_\beta^9$$

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